

# Performance of Mean-Variance & CVaR Portfolio Optimization Models in a Time of Corona Crisis

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**Cvijić, Nataša**

**Undergraduate thesis / Završni rad**

**2020**

*Degree Grantor / Ustanova koja je dodijelila akademski / stručni stupanj:* **University of Zagreb, Faculty of Economics and Business / Sveučilište u Zagrebu, Ekonomski fakultet**

*Permanent link / Trajna poveznica:* <https://urn.nsk.hr/urn:nbn:hr:148:014723>

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*Download date / Datum preuzimanja:* **2024-05-12**



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**The University of Zagreb**  
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**Performance of Mean-Variance & CVaR Portfolio Optimization Models in a Time of**  
**Corona Crisis**

# **Performance of Mean-Variance & CVaR Portfolio Optimization Models in a Time of Corona Crisis**

**Undergraduate paper**

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**Zagreb, 2020-07-11**

# **Performance of Mean-Variance & CVaR Portfolio Optimization Models in a Time of Corona Crisis**

# Abstract

Which characteristics of a portfolio are important, how can we select an optimal portfolio and which portfolio a risk-averse investor should avoid?

Pioneering theory about portfolio selection methods introduced by Harry Markowitz in the 1950's has helped to solve some of these issues in financial world. His mean-variance portfolio theory has yielded tools for the selection of efficient portfolios and is a backbone of all contemporary optimization methods. Although Markowitz's portfolio theory has faced many challenges in practice, due to some assumptions that are not mirroring the real world, it is a still basic model that underlies modern portfolio theory. The model has also been criticized because it is suitable for elliptical distributions and if returns are not elliptical, analysis can yield wrong conclusions. Due these drawbacks many risk measures have been introduced since. One of them is Value at Risk, however its sub-additivity property issues and ignorance of the worst losses in the far tail has been overcome with another risk measure Expected Shortfall or also called Conditional Value at Risk (CVaR).

This paper explores the performance of two portfolio optimization methods, Conditional Value at Risk and Mean-Variance portfolio optimization during two different periods, one before and other during the corona crisis. This approach is tested in R on a portfolio composed of four NASDAQ index stocks (Alphabet.inc (GOOGL), Tesla.inc (TSLA), Facebook.inc (FB) and Amazon.inc (AMZN)) to demonstrate whether there is a difference in portfolio performance under two different risk measures and different market conditions.

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## Chapter 1

# Introduction

### 1.1 Project Goal and Data

The aim of this thesis is to investigate two mainstream portfolio optimization methods, Markowitz mean-variance (MV) and mean-CVaR optimization method in which covariance risk is replaced by the conditional Value at Risk as risk measure. Moreover, the goal is to compare relative performance between the two optimization methods on the example of four NASDAQ composite index stocks during two different time periods. For each asset, a time series of weekly stock prices was obtained for two distinct time periods. The first time interval is from December 2018 to December 2019 and the second one is from December 2019 to July 2020. Two distinct time intervals allow better comparison and evaluation of risk measures because the chosen periods capture both positive and negative financial situations on the market. The first time interval covers a period with relative financial stability before the corona crisis, while the second time interval covers a period of corona outbreak and it exhibits big volatility and financial distress. The four stocks in the portfolio were chosen as to represent different stock-market segments. Namely, they are Google, Amazon, Facebook and Tesla. Well diversified portfolio should consist of at least 15 securities from different sectors, but due to computational simplicity here we consider just a simple case of four different securities.

### 1.2 Methodology

In this study, we examine the performance of the chosen portfolio using the MV and CVaR optimization methods. We implement optimization approaches in R software. These optimization methods could be defined using different settings: (i) computing the portfolio with the highest return for a given risk, (ii) computing the portfolio with the lowest risk for a given return, (iii) computing the portfolio with the highest return/risk ratio.

In the first case, variance as risk measure in MV and CVaR constraint cannot be the same if asset returns do not exhibit elliptical distribution. Hence, comparison would be unjustifiable and for this reason we will use the second setting minimizing the risk for a given return that allows us to compare these two optimization methods.

The same set of constraints is used to all portfolios and for both approaches, to ensure comparability in the outputs. No short sales are allowed.

### **1.3 Paper Structure**

Chapter 2 describes the core concepts of mean-variance portfolio theory and examines the criticism of this model. Distribution of returns that plays a key role in Markowitz's portfolio theory is analyzed in detail.

Chapter 3 presents components of mean-variance analysis. It discusses what constitutes portfolio's risk and return and how they are derived.

Chapter 4 introduces some of downside risk measures and discusses why CVaR is the preferred measure of risk to VaR in portfolio optimization.

Chapter 5 gives a brief introduction into basics of statistical software R. It also describes the Rmetrics framework used for portfolio selection and optimization.

Chapter 6 contains implementation and results of mean-variance optimization approach tested on a portfolio consisting of four different assets. Two cases are tested. The first one considers mean-variance analysis with restrictions set on the weights. The second one considers unlimited case of short selling with analytically closed form solution.

Chapter 7 presents implementation of CVaR in R and provides comparison of the results of MV and CVaR optimization methods.

Finally, conclusion brings important remarks and provides the prospects for further analysis of this topic.



## Chapter 2

# Markowitz Portfolio Theory

### 2.1 Background

Theories and methods developed in the field of economics have had such a great influence on a professional behavior and practice in the real world. The methods of modern investment theory are widely used by analysts and are becoming increasingly sophisticated. Understanding the investment theory can help investors capture extra returns and create new products. The beginnings of modern portfolio theory date back to 1952 when Harry Markowitz (1952) published a paper entitled “Portfolio Selection”.<sup>1</sup> His mean-variance portfolio theory provides theoretical framework for analyzing the roles of risk and return in portfolio selection. In it, Markowitz showed that the value of investment opportunities can be measured by mean return and variance of return. Also, he has developed the concept of portfolio diversification. Fundamental principle of diversification is as follows: when securities are combined into a portfolio, the resulting portfolio will have a lower level of risk than a simple average of the risks of the securities.<sup>2</sup> Furthermore, his theory yields a mechanism for the selection of portfolios of assets in a way that trades off the expected returns and the risk of potential portfolios. Given the technology available at that time, the technique was computationally very complex. However, advances in computer technology have enabled us to use sophisticated mathematical tools for solving optimization problems very easily. Optimization models play an extremely important role in financial decisions because of its wide variety of applications and the availability of efficient algorithms. Optimization is a branch of applied mathematics which refers to minimization or maximization of a given objective function subject to a certain set of constraints. A typical optimization model aims to allocate resources among possible alternative uses with the purpose of maximizing a function, for example profit. Markowitz portfolio optimization problem, also known as mean-variance optimization (MVO), can be formulated in three

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<sup>1</sup>Haugen, R. (1993). *Modern investment theory*. Englewood Cliffs, NJ: Prentice-Hall.

<sup>2</sup>Sharpe, W., Alexander, G. and Bailey, J. (2014). *Investments*. Upper Saddle River, NJ: Prentice-Hall International.

different but equivalent ways.<sup>3</sup> First, we may choose to find a minimum variance portfolio of the assets that yields a target value of expected return. Second, we may want to maximize the expected return of portfolio while limiting the variance of its return. Third, we can maximize risk-adjusted expected return.

## 2.2 Model Criticism

In spite of magnificence and importance of Markowitz's theory, many critics argue that his model is not realistic. Mean–variance portfolio optimization trades off the expected return and variability of the underlying assets assuming that the means and variances of the asset returns are known. However, in reality they are unknown and need to be estimated using historical time series. Computing efficient frontier with such estimates has produced unrealistic portfolios that have underperformed.

Mangram (2013) suggests that some of the key criticisms include<sup>4</sup>:

- „Investor ‘Irrationality’ – The assumption is that investors are rational and seek to maximize returns while minimizing risk.“ In practice investors seek highly risky assets.
- „Higher Risk = Higher Returns – The assumption that investors are only willing to accept higher amounts of risk if compensated by higher expected returns is frequently contradicted by investor's contrary actions.“
- Unlimited Access to Capital – Another key assumption is that investors have virtually unlimited borrowing capacity at a risk free interest rate
- „Efficient Markets – Markowitz' theoretical contributions to MPT are built upon the assumption that markets are perfectly efficient.“
- „No Taxes or Transaction Costs – Markowitz' theoretical contributions to MPT do not include taxes or transactions costs. “

Important criticism is a distribution normality which is not common in real world and is explained in the following section.

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<sup>3</sup>Cornuejols, G. and Tütüncü, R. (2009). *Optimization methods in finance*. Cambridge: Cambridge University Press.

<sup>4</sup>Mangram, Myles. (2013). A Simplified Perspective of the Markowitz Portfolio Theory. *Global Journal of Business Research*. 7.

## 2.3 Normal Distribution

In almost all investment decisions we use random variables. For example, the return on a security is a random variable and must be estimated. In order to deal with the uncertainty of returns, investors need to have in mind that expected return is only a one-point estimate of the entire range of possibilities. Hence, many possible returns can occur. The probability for a specific outcome is simply the chance that the stated outcome will occur. The result of considering these outcomes and their probabilities together is a probability distribution consisting of the specification of the possible outcomes that may occur and the probabilities associated with these possible outcomes.<sup>5</sup> In our analysis it is assumed that the underlying probability distribution is a normal distribution which is the most widely used probability distribution in quantitative field. It plays important role in modern portfolio theory. The defining characteristics of a normal distribution are as follows:<sup>6</sup>

- The normal distribution is completely described by two parameters: its mean  $\mu$  and variance  $\sigma^2$ . We can also define a normal distribution in terms of the mean and the standard deviation  $\sigma$ . (This is often convenient because  $\sigma$  is measured in the same units as  $\mu$ .) As a consequence, we can answer any probability question about a normal random variable if we know its mean and variance.
- The normal distribution has skewness of 0 (it is symmetric). The normal distribution has a kurtosis (measure of peakdness) of 3; its excess kurtosis (kurtosis -3.0) equals 0. As a consequence of symmetry, the mean, the median, and the mode are all equal for a normal random variable.
- A linear combination of two or more normal random variables is also normally distributed.

For a normal distribution, the probability that a particular outcome will be above or below a specified value can be determined. With one standard deviation on either side of the arithmetic mean of the distribution 68,3 % of the outcomes will be encompassed i.e. there is 68,3 % probability that the actual outcome will be within one standard deviation of the arithmetic mean.<sup>7</sup> Probabilities that the actual outcome will be within 2 or 3 standard deviations are 95% and 99 % respectively. Figure 1 represents these three cases. Possible outcomes or possible rates of the return are plotted on the horizontal axis. On the vertical axis,

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<sup>5</sup>Jones, C. (2019). *Investments*. [S.l.]: John Wiley.

<sup>6</sup>DeFusco, R., McLeavey, D., Pinto, J. and Anson, M. (n.d.). *Quantitative Investment Analysis*.

<sup>7</sup>Jones, C. (2019). *Investments*. New York: Wiley.

the probability of achieving any given rate of return is measured. The graph is shown as if the returns are continuous along the horizontal axis. Mean-variance analysis assumes that returns are normally distributed, meaning that outcomes above and below the expected value are equally likely.

However, in the real world normal distributions are rare. Many distributions are characterized by skewness different from 0 where the greater the skewness the greater is the asymmetry.

Negative skews or skewed to the left have a longer left tail and the distribution is concentrated on the right of the figure. Positive skews or skewed to the right have a longer right tail and the distribution is concentrated on the left of the figure. Return distribution that is positively skewed has frequent small losses and a few extreme gains. Return distribution that is negatively skewed has frequent small gains and a few extreme losses. Investors prefer positive skews because they dislike negative returns more than they like the same level of positive returns.

Another case in which return distribution differs from normal distribution is when the greater percentage of returns is grouped closely around the mean (being more peaked) and when there is the greater percentage of returns with large deviations from the mean (having fatter tails). The kurtosis or the peakedness of a distribution curve represents whether the possibility of extreme events is greater than a normal distribution would suggest. The more peaked the curve (leptokurtic), the greater the possibility of extreme events and vice versa, the flatter the curve (platykurtic), the less likely extreme events are to occur. Distributions with a kurtosis greater than 3 are called fat tails. Investors prefer lower kurtosis to higher kurtosis because they dislike the higher probability of losses that are associated in distributions with fat tails.

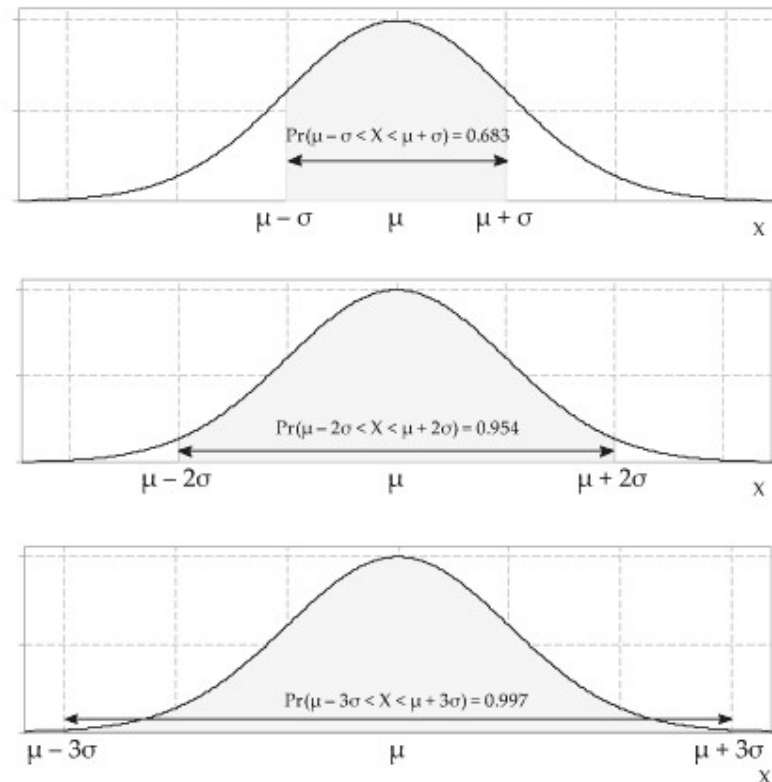


Figure 1: Three intervals of the normal distribution<sup>8</sup>

## Chapter 3

# Portfolio Environment

## 3.1 Investment

Investment can be described as the commitment of funds to one or more assets that will be held over some future time period. It refers to the management of an investor's wealth. In this paper investment refers to financial assets which are intangible assets that represent a claim on future cash flows. Examples of financial assets are: equities, bonds, derivatives, funds. Each asset is characterized by following statistics presented in more details further in the paper: the expected return on asset,  $E(r_i)$  or  $\mu_i$ , the variance of asset  $i$ 's return,  $(\sigma_i)$ , the covariance of asset  $i$ 's and asset  $j$ 's returns  $(\sigma_{ij})$ .

<sup>8</sup>Amsi.org.au.(2020). *Content-Normal\_distribution*. [online] Available at: [https://amsi.org.au/ESA\\_Senior\\_Years/SeniorTopic4/4f/4f\\_2content\\_3.html](https://amsi.org.au/ESA_Senior_Years/SeniorTopic4/4f/4f_2content_3.html)

Although individual asset returns and risks are important, total portfolio return and risk is what matters, because risk can be reduced by packaging different financial assets together to form a portfolio. Hence, in analyzing investment's returns and risk, we should observe total portfolio held by investor.

Suppose there is an investor who wants to invest certain amount of money in various financial assets with random returns. For each financial asset  $i$ ,  $i=1,2,\dots,n$ , estimation of its expected return  $E(r_i)$  and variance  $\sigma_i^2$  is calculated using historical returns data

### 3.2 Return and Expected Return

Investors wish to earn a return on their money. Return, from the aspect of portfolio theory, is a gain of a financial asset in a particular period. Individuals who buy riskier financial assets are trying to earn returns larger than those available from safer assets.

For compound interest, return  $r_t$  over some unit period  $t$  is equal to:

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}} \quad (1)$$

Where  $P_t$  is the price of the asset at the end of the period  $t$  and  $P_{t-1}$  is the price of the asset at the beginning of the period  $t$ . However, in our analysis we use continuously compounded return on the security which treats time as continues. Continuously compounded return associated with a holding period is the natural logarithm of the ending price over the beginning price:

$$r_t = \ln\left(\frac{P_t}{P_{t-1}}\right) \quad (2)$$

As it is asserted, investors invest because they expect to achieve some gain. However at the end of the investing period, they are left with actual or realized return. Investors' realized return may turn out to be higher or lower than the expected return. The Markowitz model provides us with efficient portfolios assuming that we have perfect information on the expected return ( $\mu_i$ ). Hence, it is important to make a good estimation of the expected return  $\mu_i$ . We can estimate it by using time series of past returns. Under assumption that past returns somewhat reflect future returns, we assume that average of historical data represents expected return of each security.

Given a portfolio with  $n$  securities, the expected portfolio return over a specific time period is the weighted average of the expected returns on the individual assets in the portfolio:

$$E(r_p) = \sum_{i=1}^n x_i E(r_i) = \mathbf{E}(\mathbf{r})^T \mathbf{x} = \mathbf{x}^T \mathbf{E}(\mathbf{r}) \quad (3)$$

Where

$x_i$  = the portfolio weight for the  $i$ th asset

$E(r_i)$  = the expected return on the  $i$ th asset

$$\sum_{i=1}^n x_i = 1 \quad (4)$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \text{ is the vector of portfolio proportions} \quad (5)$$

$\mathbf{x}^T$  is transpose of this vector,  $\mathbf{x}^T = [x_1, x_2, x_3, x_4, \dots, x_n]$

$$\mathbf{E}_r = \begin{bmatrix} E_{r_1} \\ E_{r_2} \\ E_{r_3} \\ \vdots \\ E_{r_n} \end{bmatrix} \text{ is the vector of means } \mathbf{E}(\mathbf{r}) \quad (6)$$

$\mathbf{E}(\mathbf{r})^T$  is transpose of this vector, that is  $\mathbf{E}(\mathbf{r})^T = [E(r_1), E(r_2), E(r_2), \dots, E(r_n)]$

### 3.3 Risk

Investors should be able to quantify risk. The risk associated with an investment is a chance that the actual outcome of an investment will differ from the expected return. Hence, we can say that risk arises out of variability or dispersion in the likely outcomes. For example, if asset's return has no variability, it is assumed it has no risk. The most commonly used measure of dispersion is variance or standard deviation. The greater the standard deviation or the variance, the greater the dispersion is. The variance of returns is calculated by computing the deviation of each observed return from the average returns and then squaring each deviation. The sum of these squared deviations divided by the number of observations minus one is the sample variance. We are using sample variance because we are dealing only with the subset of outcomes for the random variable. If we were dealing with the complete set of all outcomes for a given random variable, then we would consider the sum of the squared deviations would be divided by the number of observations, which is called population variance. The standard deviation is the square root of the variance.

The formula for standard deviation is

$$\sigma^2 = \frac{\sum_1^n (x_i - \bar{x})^2}{n - 1}$$

Where

$\sigma^2$  = the variance of a set of observations

$x_i$  = the  $i$ th observation in the set

$\bar{x}$  = the mean of the observations

$n$  = the number of observations in the set

$\sigma$  = standard deviation

It is at this point that the basis of modern portfolio theory emerges, which can be stated as follows: although the expected return of a portfolio is a weighted average of its expected returns, portfolio risk (as measured by the variance or standard deviation) is not a weighted average of the risk of the individual securities in the portfolio.<sup>9</sup> Why is it so? Because investors can reduce the portfolio risk beyond what would be a weighted average of the individual assets' risk.

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<sup>9</sup>JONES, C. (2019). *INVESTMENTS*. [S.l.]: JOHN WILEY.



Statistical concepts that measure the relationship between returns of one asset and another are covariance and correlation coefficient. Covariance is an absolute measure of the degree of association between the returns for a pair of assets.<sup>10</sup> It measures the extent to which two or more random variables move together over time. If the returns are positively related to one another, their covariance will be positive. And vice versa: if there is no relationship between them, the covariance should be zero. In calculating covariances it does not matter which variable comes first;  $\sigma_{ij} = \sigma_{ji}$ . The covariance of one variable with itself is the same as the variance of that variable:  $\sigma_{ii} = \sigma_i^2$ . What effect the covariance has on portfolio variance? For example, in case of two securities, if one asset tends to have high returns when the other tends to have low returns (relative to its expected return) then the returns on one security tend to offset the returns on the other, decreasing the variability of returns in the portfolio. Covariance is often represented in a covariance matrix.

Given two random variables  $R_i$  and  $R_j$ , the covariance between them is equal to

$$Cov(R_i, R_j) \text{ or } \sigma_{ij} = E \left[ (R_i - E(R_i)) (R_j - E(R_j)) \right] \quad (7)$$

where

$\sigma_{ij}$  = the covariance between asset  $i$  and asset  $j$

$R_i$  = possible return on asset  $i$

$ER_i$  = the expected value of the return on security  $i$

Equation 8 states that covariance between two random variables is the probability-weighted average of the cross-products of each random variables deviation from its own expected value.<sup>11</sup>

Although covariance is critical input in defining the variance of a portfolio it does not entirely describe the relationship between the two assets. It is unbounded and its range extends from minus to plus infinity. However, we can standardize covariance by dividing it by the product of the standard deviations for the two assets and thus get a better descriptor called correlation coefficient. Correlation coefficient is a relative measure of co-movements between asset

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<sup>10</sup>JONES, C. (2019). *INVESTMENTS*. [S.I.]: JOHN WILEY.

<sup>11</sup>JONES, C. (2019). *INVESTMENTS*. [S.I.]: JOHN WILEY.

returns. Its values fall into interval from -1 to 1. Expression for correlation coefficient between two variables is as follows:

$$\rho_{XY} = \frac{\sigma_{xy}}{\sigma_x \sigma_y} \quad (8)$$

Where

$\sigma_{xy}$  = covariance between securitise x and y

$\sigma_x$  = standard deviation of x

$\sigma_y$  = standard deviation of y

When  $\rho_{XY}$  is equal to +1,0 securities are perfectly positively correlated and there is no reduction in portfolio risk. Two securities move perfectly in the same direction. When  $\rho_{XY} = 0$  portfolio risk is reduced. When  $\rho_{XY}$  is -1.0, portfolio risk could be eliminated by combining these two perfectly negatively correlated securities.. In the real world extreme correlations are rare and securities typically have positive correlation to each other. Investors prefer to find securities with the least positive correlation possible, preferably with negative correlation. But in general, they will be faced with positively correlated security returns.<sup>12</sup> In his portfolio theory Markowitz recognized the importance of the variances and covariances. Up to this point we have seen that portfolio risk is determined by three factors: the variance of each security, the covariances between securities and the portfolio weights for each security. Hence formula for portfolio variance is

$$\sigma_p^2 = \sum_{i=1}^n \sum_{j=1}^n x_i x_j \sigma_{ij} = \sum_{i=1}^n x_i^2 \sigma_i^2 + 2 \sum_{i=1}^n \sum_{j=1}^n x_i x_j \sigma_{ij} \quad (9)$$

where

$\sigma_p^2$  is the variance of the return on the portfolio

$\sigma_i^2$  is the variance of return for security  $i$

$\sigma_{ij}$  is the covariance between the returns for securities  $i$  and  $j$

$x_i$  are the portfolio weights.

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<sup>12</sup>JONES, C. (2019). *INVESTMENTS*. [S.l.]: JOHN WILEY

As the number of securities in the portfolio increases, the importance of individual securities variances decreases, while the importance of the covariance increases.<sup>13</sup>

Consider the first term of the equation 9 and assume that an investor invested equal amounts in each security. The weights will be  $1/n$ . Substituting it in equation (9) we get:

$$\sum_{i=1}^n [(1/n)^2] \sigma_i^2 = \frac{1}{n} \sum_{i=1}^n \sigma_i^2 / n \quad (10)$$

The term in brackets represents an average variance for the assets in the portfolio. As  $n$  becomes larger, average variance gets smaller, approaching zero for large values. Hence, the risk of well-diversified portfolio will greatly depend on the second term of equation representing the covariance relationship.

Let  $\Sigma$  denote the  $n \times n$  symmetric covariance matrix with  $\sigma_{ii} = \sigma_i^2$  and  $\sigma_{ji} = \rho_{ij} \sigma_i \sigma_j$  for  $i \neq j$ . Then the variance of portfolios return in a matrix form can be written as

$$\sigma_p^2 = \sum_{i=1}^n \sum_{j=1}^n x_i x_j \sigma_{ij} = \mathbf{x}^T \Sigma \mathbf{x} \quad (11)$$

### 3.4 Mean-Variance Analysis

Mean-variance portfolio theory, the oldest and perhaps the most accepted part of modern portfolio theory, provides the theoretical foundation for examining the roles of risk and return in portfolio selection. Mean-variance optimization (MVO) is based on the following assumptions:<sup>14</sup>

- All investors are risk-averse, meaning they prefer less risk to more for the same amount of expected return.
- Expected returns for all assets are known.
- The variances and covariances of all asset returns are known.
- There are no transaction costs or taxes.

Investors do not have the same tolerance for risk, but risk-averse investors prefer as little risk as possible for a given level of expected return. In the real world, expected returns and variances of returns are not known but estimated, which presents a drawback of mean-variance analysis but we will assume that estimated returns reflect future returns. Investor's objective in the mean-variance analysis of portfolio selection is to find an efficient portfolio.

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<sup>13</sup>JONES, C. (2019). *INVESTMENTS*. [S.I.]: JOHN WILEY.

<sup>14</sup>DeFusco, R., McLeavey, D., Pinto, J. and Anson, M. (n.d.). *Quantitative Investment Analysis*.

Markowitz was the first to derive the concept of efficient portfolio, which is one that offers the highest expected return for a given level of risk. Hence, when an investor quantifies his tolerance for risk using standard deviation, he wants the portfolio that will yield the greatest possible return for his quantified tolerance for risk. To start identifying efficient portfolios, we must explain the following concepts, plotted in the figure 2. Assets in the figure 2 yield the feasible set of portfolios. The feasible set is the set of all portfolios that can be formed from a group of  $n$  assets. Feasible set is the area inside and to the right of the curved line. An feasible portfolio is called an envelope portfolio if for a given level of expected return it has minimum variance. Minimum-variance set or envelope (different names are used in various literatures) is the set of all envelope portfolios. Envelope is the border of the feasible region and it is represented by the curved line plotted in two-dimensional graph representing portfolios that have minimum variance for each given level of expected return. If we move to the right from a point which represents a portfolio on the envelope, we reach other portfolios. They have greater risk for the same level of expected return. An envelope portfolio minimizes variance for a given targeted return. Finally, efficient portfolio is the one that has the highest expected return among all portfolios with the same variance. The set of all efficient portfolios is called the efficient frontier.

An efficient portfolio maximizes expected return for a given level of risk and is represented as bold curve, which is the upper part of the envelope set in the figure 2. It is important to emphasize that it is impossible to find portfolios that are located above the efficient frontier

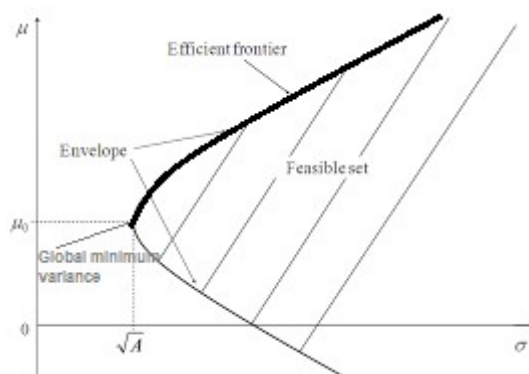


Figure 2:Efficient Frontier

Markowitz's portfolio optimization model can be formulated in three different but equivalent ways<sup>15</sup>. In this paper we will solve the problem of finding a minimum variance portfolio given the target value of expected return. It can be stated as the following quadratic optimization problem with linear constraints:

$$\min \mathbf{x}^T \Sigma \mathbf{x} \quad (12)$$

s.t.

$$\mathbf{x}^T \mathbf{1} = 1 \quad (13)$$

$$\mathbf{x}^T \mathbf{E}(\mathbf{r}) \geq \bar{r} \quad (14)$$

$$\mathbf{x} \geq 0 \quad (15)$$

The objective function (12) is the variance of the portfolio that we want to minimize,  $\sigma_p^2 = \mathbf{x}^T \Sigma \mathbf{x}$ . The first constraint (13) denotes that we are considering only feasible portfolios, meaning that the available capital is fully invested and the sum of weights is equal to 1. The second constraint (14) denotes that the expected return should be no less than the target value  $\bar{r}$ . Third constraint (15) forbids short selling, that is all weights should be non-negative. Short selling is the ability to sell an asset that is not owned by the investor, i.e. borrowing it from someone else and then selling it, with the obligation of buying back the asset and returning it to the original owner. In the case when short selling is allowed, Markowitz's portfolio model has unique solution and can be solved analytically. However, if the weights are non-negative, which forbids short selling, then the optimization has to be done numerically.<sup>16</sup> In this analysis, two cases of MVO problem will be solved. In the first case, we solve the model which allows short sales, that is model (12-14). This model will be solved by using the Lagrangian multiplier method and results will be implemented in R. The second model considers the case when short sales are not allowed, that is model (12-15). The model will be solved using a quadratic solver in R, thus obtaining weights of the portfolios. The R package `quadprog` provides the function `solve.QP()` for solving quadratic programming problems in which variance is minimized subject to some set of linear constraints.

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<sup>15</sup>Cornuejols, G. and Tütüncü, R. (2007). *Optimization methods in finance*. Cambridge: Cambridge University Press.

<sup>16</sup>Würtz, D., Setz T., Chalabi, Y., (2011). Portfolio Optimization with R/Rmetrics .

## Chapter 4

### Mean-CVaR Portfolio Theory and Settings

Mean-Variance analysis assumes that asset returns follow normal distribution. However, these assumptions do not mirror the real world. Many assets can have skewed and leptokurtic return distributions. Moreover, Markowitz approach is not suited to capture events of low probability such as default risk. Because of these shortcomings of the variance measure, many researchers aimed to develop models that take into account asymmetries in the asset returns. In 1994 the concept of Value at Risk (VaR) was introduced as an alternative to the volatility as traditional measure of the risk. VaR has become very popular method in financial practice for measuring risk since investors are concerned about the maximal loss of a portfolio over specific time interval at a defined confidence level.

VaR is the largest likely loss from market risk (expressed in currency units) that an asset or portfolio will suffer over time interval with a degree of certainty selected by the user<sup>17</sup>

However, VaR exhibits many theoretical and practical shortcomings. One of them is that VaR ignores the severity of losses beyond the cutoff. There is no information about the extent of the tail distributions, which means that portfolio return can yield lower values than VaR, resulting in severe losses. More importantly, VaR is not a coherent risk measure in the sense defined by Artzner, et al. (1999)<sup>18</sup> Coherent risk measure satisfies four axioms and VaR does not fulfill subadditivity axiom and hence ignores the concept of diversification. Therefore, portfolio's VaR can be larger than the sum of the VaRs of its components. Moreover, when asset returns are expressed in terms of discrete distributions, VaR is non-convex and discontinuous function of the portfolio positions and exhibits multiple local extremes<sup>19</sup> Therefore, it makes optimization process very complex, difficult and impractical to use. In this regard, suitable risk measures have been investigated. Artzner et al. (1997) introduced certain criteria and properties an ideal risk measure needs to satisfy in order to be coherent. These are as follows. Let  $X$  be a set of outcomes and let  $p(X)$  be a measure of risk.

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<sup>17</sup> Choudhry, M., 2013. An Introduction To Value-At-Risk. New York: Wiley.

<sup>18</sup> Dempster, M., Mitra, G. and Pflug, G., 2009. Quantitative Fund Management. Boca Raton: CRC Press.

<sup>19</sup> Uryasev, S., 2020. [online] Ise.ufl.edu. Available at: <[https://www.ise.ufl.edu/uryasev/files/2011/11/VaR\\_vs\\_CVaR\\_CARISMA\\_conference\\_2010.pdf](https://www.ise.ufl.edu/uryasev/files/2011/11/VaR_vs_CVaR_CARISMA_conference_2010.pdf)> [Accessed 7 July 2020].

- **Axiom 1: Subadditivity**

By adding two portfolios together, the total risk cannot worsen than adding the two risk separately, meaning that diversification have a reduction in risk.

$$p(X+Y) \leq p(X) + p(Y)$$

- **Axiom 2: Monotonicity**

If  $X < Y$ , then  $p(X) > p(Y)$

If one portfolio has better values than another then its risk will be better.

- **Axiom 3: Homogeneity**

For all  $X > 0, p(X) = Xp(1)$

Increasing your portfolio should double your risk by the same factor

- **Axiom 4: Translation invariance**

$$p(X+c) = p(X) - c$$

By adding cash to your portfolio this should simply reduce its risk by constant  $c$ .

If a risk measure does not satisfy one of the four axioms, it can lead to incorrect results and wrong estimation of risk. In order to enhance the VaR to meet subadditivity as well as the other three properties and be a coherent risk measure, Rockafellar and Uryasev (2002) introduced Expected Shortfall (ES) or Conditional Value at Risk. The CVaR risk has been proved by Pflug (2000) and Rockafellar to be a coherent risk. Uryasev has shown that CVaR is a continuous and convex function of the portfolio components.

CVaR is more sensitive than VaR to the tail behavior of loss distribution because it quantifies the mass in the tail of the distribution exceeding VaR. Simply put, it is a conditional expectation of the loss above the VaR for a given time interval and confidence level. Although CVaR has not become norm in finance industry, CVaR is gaining importance in the insurance industry.<sup>20</sup> According to the Rockafellar and Uryasev, minimization of CVaR also yields similar optimal solutions in VaR terms because CVaR measure will always be greater or equal to the VaR measure for a given portfolio. Hence, portfolio with low CVaR will also have low VaR as well. Furthermore, in case of normal distribution, these two measures are the

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<sup>20</sup>Krokhmal, P., Uryasev, t. and Palmquist, J., 2001. Portfolio optimization with conditional value-at-risk objective and constraints. *The Journal of Risk*, 4(2), pp.43-68.

same yielding the identical optimal portfolio. However, problem arises for much skewed distributions where CVaR and VaR can be entirely different. Besides easiness of use optimization methods and the differences in mathematical properties, the reasons behind the choice between CVaR and VaR are also grounded on acceptance by regulators, reliability of statistical estimation etc.

The reasons why CVaR could be considered more consistent measure than VaR can be summarized as follows:

- It has better mathematical properties than VaR due its coherence. CVaR is a continuous and convex function of its portfolio components, whereas VaR could exhibit discontinuity.
- CVaR can be solved using linear programming algorithms which allows the optimization of very large problems, whereas VaR is difficult to optimize due to its non-convexity and complexity.
- VaR risk measure ignores the scenarios beyond the VaR, whereas CvaR accounts for it, that is it is more conservative (which can be both good and bad, depending on one's goals).

#### 4.1 Mathematical Formulation of CVaR Optimization Problem

For a specified confidence level  $\alpha$ , CVaR is defined as follows<sup>21</sup>:

Let  $f(\mathbf{w}, r)$  be the loss function of a portfolio  $\mathbf{w}$  from the set  $X$  of feasible portfolios and let  $r$  be a realization of random events. Furthermore, let  $p(r)$  be a probability density function of the random variable  $r$  and let  $\psi$  be the cumulative distribution function for the loss related to fixed portfolio  $\mathbf{w}$ . Function  $\psi(\mathbf{w}, \gamma)$  is a continuous function which is also non-decreasing with respect to  $\gamma$ , where  $\gamma$  is the threshold value loss. The function  $\psi(\mathbf{w}, \gamma)$  can be interpreted as the probability that the losses do not exceed threshold  $\gamma$  and it is defined as:

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<sup>21</sup>Würtz, D., Setz T., Chalabi, Y., (2011). Portfolio Optimization with R/Rmetrics



$$\psi(\mathbf{w}, \gamma) = \int_{f(\mathbf{w}, r) \leq \gamma} p(r) dr$$

For a given confidence level  $\alpha$ , the  $Var_\alpha$  associated with portfolio  $\mathbf{w}$  is defined as

$$VaR_\alpha(\mathbf{w}) = \min\{\gamma \in \mathbb{R} : \psi(\mathbf{w}, \gamma) \geq \alpha\}$$

Now, the  $CVaR_\alpha$  is defined as

$$CVaR_\alpha(\mathbf{w}) = \frac{1}{1 - \alpha} \int_{f(\mathbf{w}, r) \leq VaR_\alpha(\mathbf{w})} f(\mathbf{w}, r) p(r) dr$$

The problem of determining mean-CVaR portfolio can be stated as follows:

$$\min CVaR_\alpha(\mathbf{w})$$

s.t.

$$\mathbf{w}^T \boldsymbol{\mu} = r$$

$$\mathbf{w}^T \mathbf{1} = 1$$

where  $\boldsymbol{\mu}$  is the vector that estimates the expected mean of the assets and  $r$  is target return. Minimizing  $Var_\alpha$  and  $CVaR_\alpha$  is not the same and because the definition of  $CVaR_\alpha$  includes the  $Var_\alpha$  it is difficult to optimize this function. Instead, the following auxiliary function is considered:

$$F_\alpha(\mathbf{w}, \gamma) = \gamma + \frac{1}{1 - \alpha} \int_{f(\mathbf{w}, r) \geq \gamma} (f(\mathbf{w}, r) - \gamma) p(r) dr$$

Alternatively, it can be written as

$$F_\alpha(\mathbf{w}, \gamma) = \gamma + \frac{1}{1 - \alpha} \int (f(\mathbf{w}, r) - \gamma)^+ p(r) dr$$

Where  $z^+ = \max(z, 0)$ .

This final function of  $\gamma$  has the following properties useful for the computation of  $Var_\alpha$  and  $CVaR_\alpha$ :

- $F_\alpha(\mathbf{w}, \gamma)$  is a convex function of  $\gamma$
- $VaR_\alpha(\mathbf{w})$  is minimizer of  $F_\alpha(\mathbf{w}, \gamma)$
- The minimum value of the function  $F_\alpha(\mathbf{w}, \gamma)$  is  $CVaR_\alpha(\mathbf{w})$

$CVaR_\alpha$  can be optimized via optimization of function  $F_\alpha(\mathbf{w}, \gamma)$  with respect to the weights  $\mathbf{w}$  and  $\gamma$  equal to VaR. If the loss function  $f(\mathbf{w}, r)$  is a convex function of the portfolio variables  $\mathbf{w}$ , then  $F_\alpha(\mathbf{w}, \gamma)$  is also a convex function of  $w$ . Hence, feasible portfolio set  $W$  is also convex. Due to these properties, optimization problems are smooth convex optimization problems than can be easily solved with well-known optimization techniques.

## Chapter 5

# R

## 5.1 About R

R is a program for statistical computing and graphics. It is a GNU project which is similar to the S language and environment which was developed at Bell Laboratories (formerly AT&T, now Lucent Technologies) by John Chambers and colleagues. R can be considered as a different implementation of S. There are some important differences, but much code written for S runs unaltered under R. R is an open source program, meaning that it can be upgraded and changed with new versions or by users themselves. “R provides a wide variety of statistical (linear and nonlinear modeling, classical statistical tests, time-series analysis, classification, clustering, ...) and graphical techniques, and is highly extensible.”<sup>22</sup>. R-studio is integrated development environment that simplifies usage of R. Its interface is organized so that the user can clearly view graphs, data tables, R code and output all at the same time. It also offers an Import-Wizard-like feature that allows users to import CSV, Excel, SAS (\*.sas7bdat), SPSS (\*.sav), and Stata (\*.dta) files into R without having to write the code to

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<sup>22</sup>available at: <https://www.r-project.org/about.html>, accessed: 27.12.2019.

do so. “<sup>23</sup>R-studio includes a console, syntax-highlighting editor that supports direct code execution, as well as tools for plotting, history, debugging and workspace management.<sup>24</sup>When user launches RStudio, she/he sees the following four windows or panes.1) Source2) Console 3) Environment/History 4) Files/Plots/Packages/Help<sup>25</sup> . Figure 2 shows windows in RStudio.

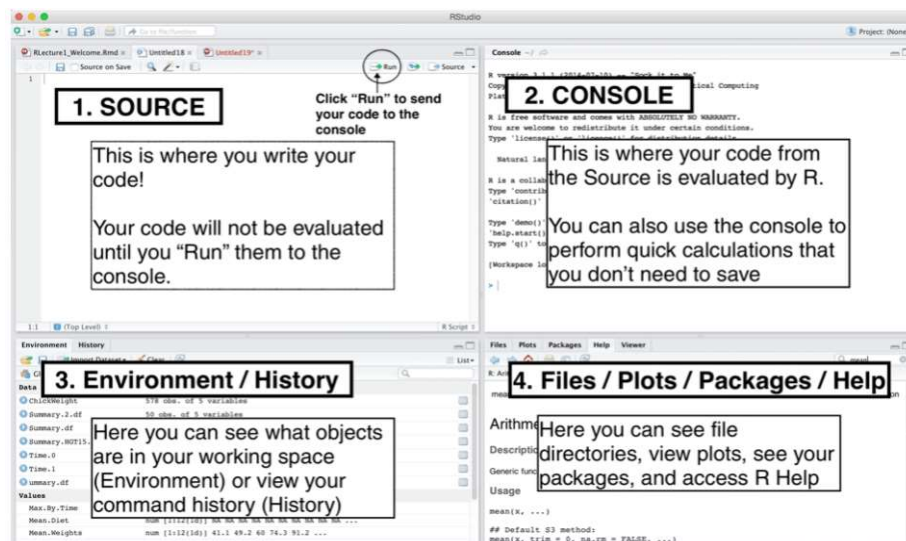


Figure 3: Windows in Rstudio

Source: <https://datascienceplus.com/introduction-to-rstudio/> accessed: 27.12.2019

R is very useful for portfolio management because it offers many useful financial packages. R packages are a collection of R functions, compiled code and sample data. They are stored under a directory called "library" in the R environment. By default, R installs a set of packages during installation. More packages are added later, when they are needed for some specific purpose. When we start the R console, only the default packages are available by default“<sup>26</sup>.

<sup>23</sup>available at: <https://www.r-project.org/about.html>, accessed: 27.12.2019.

<sup>24</sup> Available at: <https://datascienceplus.com/introduction-to-rstudio/>, accessed: 27.12.2019

<sup>25</sup>Available at: <https://datascienceplus.com/introduction-to-rstudio/>, accessed: 27.12.2019

<sup>26</sup>available at: [https://www.tutorialspoint.com/r/r\\_packages.htm](https://www.tutorialspoint.com/r/r_packages.htm), accessed:29.12.2019

## 5.2 Portfolio Functions in R

Portfolio construction and optimization with the `RmetricsfPortfolio` package relies on four pillars<sup>27</sup>:

- Definition of the portfolio input, writing specifications, loading the data of the assets, and setting up the constraints.
- Optimization of the portfolio, including the computation of single portfolios such as feasible, efficient, tangency (max reward/risk) or minimum variance (global minimum risk) portfolios, and the evaluation of the entire efficient frontier.
- Generation of portfolio reports: printing, plotting and summarizing the results.
- Analysis of portfolio performance, including rolling analysis, backtesting and benchmarking.

R packages that are used for portfolio and financial analysis in this paper are: `fPortfolio`, `PerformanceAnalytics`, `quantmod`, `ggplot2`, `plotrix`, `timeSeries`.

In order to implement portfolio optimization, we need to specify the model we want to use, then choose the corresponding portfolio settings and finally the type of programming solver that should be applied in optimization (linear, quadratic or nonlinear). All parameter settings that specify portfolio are described by a class `PFOLIOSPEC`. This portfolio specification class consists of following slots: the `model`, the `portfolio`, the `optim` and the `message` slot. The first slot `model` carries the information about the type of the portfolio (MV-mean variance (Markowitz) portfolio, CVAR-mean conditional Value at Risk portfolio etc.), the objective function needed to be optimized and the estimator for mean and covariance. The second slot `portfolio` holds all parameters needed to be specified like the weights, the target return and risk, the number of frontier points. The third slot `optim` includes the information about the solver that should be used (`solveRquadprog`, `soleRglpk`..). The last slot `message` has a list of optional messages.

Default setting of the function `portfolioSpec()` is the mean-variance Markowitz portfolio specification with long-only constraints. The solver used for optimization is a quadratic programming solver (QP); `solveRquadprog`.

Printed output represents list of the arguments for the default settings of `portfolioSpec()` that contains the same arguments as the specification for mean-variance model.

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<sup>27</sup>Portfolio Optimization with R/Rmetrics (D. Würtz, T. Setz, Y. Chalabi, W. Chen, A. Ellis)

```

>formals(portfolioSpec)
$model
list(type = "MV", optimize = "minRisk", estimator = "covEstimator",
tailRisk = list(), params = list(alpha = 0.05))

$portfolio
list(weights = NULL, targetReturn = NULL, targetRisk = NULL,
riskFreeRate = 0, nFrontierPoints = 50, status = NA)

$optim
list(solver = "solverQuadprog", objective = c("portfolioObjective",
"portfolioReturn", "portfolioRisk"), options = list(meq = 2),
control = list(), trace = FALSE)

$messages
list(messages = FALSE, note = "")

$ampl
list(ampl = FALSE, project = "ampl", solver = "ipopt",
protocol = FALSE, trace = FALSE)

> print(MV)

Model List:
Type:                MV
Optimize:             minRisk
Estimator:           covEstimator
Params:              alpha = 0.05

Portfolio List:
Target Weights:      NULL
Target Return:       NULL
Target Risk:         NULL
Risk-Free Rate:      0
NumberOfFrontierPoints: 50

Optim List:
Solver:              solverQuadprog
Objective:           portfolioObjectiveportfolioReturnportfolioRisk
Options:             meq = 2
Trace:              FALSE

```

## Chapter 6

# Portfolio Optimization with Rmetrics

## 6.1. Introducing Portfolio Components

In this section we compute a minimum-variance set off our stocks with large trading volume from NASDAQ index. We consider the following stocks: Alphabet.inc (GOOGL), Tesla.inc (TSLA), Facebook.inc (FB) and Amazon.com.inc (AMZN) Selection of these stocks is based on the growing industries such as artificial intelligence/internet retail, electric cars, and Internet Content & Information. Figure 4 shows candlestick charts that describe price

movements of selected stocks in the period from October 2019 to December 2019 (time period of 10 weeks). Candlesticks are composed of the body (green or orange) that illustrates the opening and closing trades and an upper and a lower shadow (wick) that represents the highest and lowest traded prices of a stock during the time interval represented. Advantage of using this chart type with color customized candles over the other chart types is that it can be easily understood. A green candle is a bullish candle and an orange candle is a bearish candle.

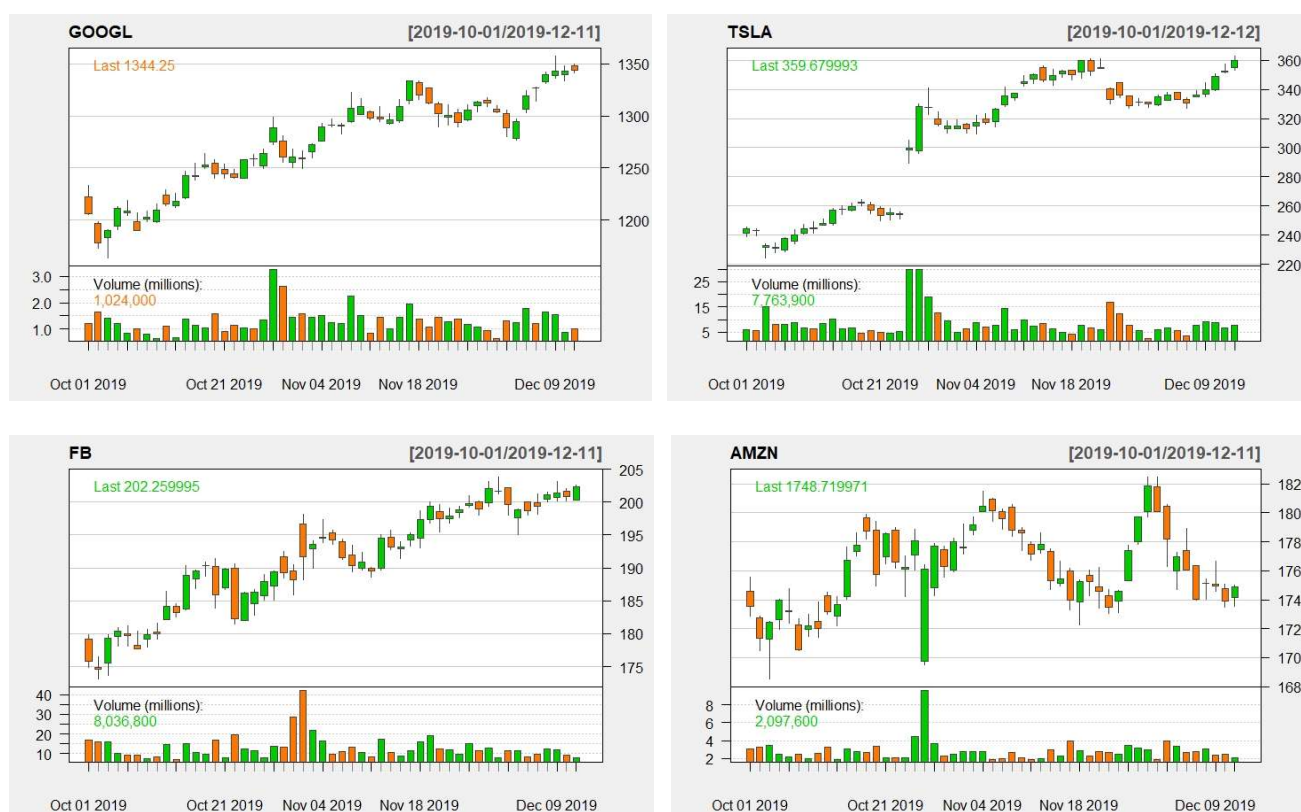


Figure 4: Candlestick charts of portfolio stocks

Source: Authors calculation (R-studio)

The data used in this section are secondary data in the form of time series data of weekly stock prices in the period from December 2018 to December 2019 (52 weeks). First we upload data using a package `quantmod` and its function `getSymbols` that downloads financial data straight into R from Yahoo website. Secondly, the continuously compounded returns are calculated using `ROC` function which calculates the rate of change of a series over  $n$  periods. Finally, the data were converted into timeSeriesobjects using function `as.timeSeries`.

```
>tickers<- c("FB", "AMZN", "GOOGL", "TSLA")
>portfolioPrices<- NULL
> for(tickerintickers) {
+   portfolioPrices<- cbind(portfolioPrices,
```

```
+ getSymbols.yahoo(ticker, from='2018-12-12',to='2019-12-12',
periodicity = 'weekly', auto.assign=FALSE)[,4])
+ }
>portfolioReturns<- na.omit(ROC(portfolioPrices))
>portfolioReturns<- as.timeSeries(portfolioReturns)

>portfolioReturns
```

```
GMT
FB.CloseAMZN.CloseGOOGL.CloseTSLA.Close
2018-12-17 -0.142316232 -0.144700646 -0.059205880 -0.1342386920
2018-12-24 0.063938102 0.070469440 0.054411808 0.0431497277
2018-12-31 0.035039543 0.063799502 0.029549054 -0.0496756098
2019-01-07 0.041532187 0.040534813 -0.012695360 0.0889977369
2019-01-14 0.042478412 0.033352743 0.039447669 -0.1387862005
2019-01-21 -0.006888494 -0.015225573 -0.005242689 -0.0174207671
2019-01-28 0.106225931 -0.026900413 0.015413803 0.0498091693
2019-02-04 0.009728608 -0.023650566 -0.014624295 -0.0207447552
2019-02-11 -0.029289921 0.012346171 0.015526795 0.0067788587
2019-02-18 -0.003760916 0.014576603 -0.002745695 -0.0437183215
2019-02-25 0.002406146 0.024322358 0.028221591 0.0002714775
2019-03-04 0.044119527 -0.030939122 0.001261655 -0.0367961417
2019-03-11 -0.021575484 0.054952642 0.034469587 -0.0311336398
2019-03-18 -0.009929847 0.030147854 0.014470925 -0.0403788248
2019-03-25 0.014198384 0.009014242 -0.025800964 0.0563347619
2019-04-01 0.052756013 0.031251609 0.028942570 -0.0176638247
2019-04-08 0.019052518 0.003141033 0.009268099 -0.0267586045
2019-04-15 -0.004589000 0.010057380 0.015210091 0.0205567642
2019-04-22 0.071480272 0.046667754 0.028546315 -0.1502426823
2019-04-29 0.020571306 0.006046369 -0.071267328 0.0812001042
2019-05-06 -0.037158101 -0.037632532 -0.018590495 -0.0627442440
2019-05-13 -0.016272669 -0.011162708 0.000975864 -0.1266366386
2019-05-20 -0.023147692 -0.024766436 -0.026152288 -0.1016658960
2019-05-27 -0.020026871 -0.026797264 -0.028606327 -0.0291140650
2019-06-03 -0.023488879 0.016183239 -0.035067763 0.0993476377
2019-06-10 0.045005880 0.035738901 0.016643354 0.0496978811
2019-06-17 0.052687552 0.022021696 0.035334394 0.0317807003
2019-06-24 0.009684050 -0.009288040 -0.038561542 0.0071859016
2019-07-01 0.017463176 0.025691242 0.045027392 0.0422351133
2019-07-08 0.042222242 0.034445244 0.011123790 0.0501171242
2019-07-15 -0.032292041 -0.023384160 -0.012113089 0.0520722929
2019-07-22 0.006983018 -0.010989022 0.095723780 -0.1241359675
2019-07-29 -0.055213734 -0.063643796 -0.040062065 0.0272520137
2019-08-05 -0.006209047 -0.008626223 -0.006221603 0.0028550100
2019-08-12 -0.022339828 -0.008338596 -0.008183840 -0.0662732542
2019-08-19 -0.032925907 -0.024251696 -0.021974577 -0.0396027459
2019-08-26 0.043592825 0.015128323 0.031528498 0.0650558043
2019-09-02 0.009754644 0.031705227 0.013175748 0.0081225715
2019-09-09 -0.001601383 0.003174625 0.027561238 0.0751437710
2019-09-16 0.014531392 -0.024869828 -0.008251544 -0.0188552870
2019-09-23 -0.069940968 -0.039049107 -0.003168038 0.0062558878
2019-09-30 0.018739136 0.008196100 -0.012302612 -0.0451973709
2019-10-07 0.020514130 -0.004453312 0.003914835 0.0687076736
2019-10-14 0.008972085 0.014667395 0.023333313 0.0358964648
2019-10-21 0.010916751 0.002171140 0.015857099 0.2445283264
2019-10-28 0.030040770 0.016950552 0.006268338 -0.0462167772
2019-11-04 -0.014462090 -0.003108439 0.028476500 0.0733052095
2019-11-11 0.022076919 -0.026319333 0.018573603 0.0436157316
2019-11-18 0.018887650 0.003575100 -0.030353915 -0.0558514206
2019-11-25 0.014083996 0.031063976 0.008022279 -0.0093518041
2019-12-02 -0.002930276 -0.027701396 0.026708844 0.0178729443
2019-12-09 0.006000325 -0.001645567 0.003621939 0.0488341012
```

Internal data structure of the portfolio looks as follows.

```
>portfolioData<-portfolioData(data=portfolioReturns, spec=portfoliospec())
> print(portfolioData)
```

Head/TailSeries Data:

```
GMT
FB.CloseAMZN.CloseGOOGL.CloseTSLA.Close
2018-12-17 -0.14231623 -0.14470065 -0.05920588 -0.13423869
2018-12-24 0.06393810 0.07046944 0.05441181 0.04314973
2018-12-31 0.03503954 0.06379950 0.02954905 -0.04967561
GMT
FB.CloseAMZN.CloseGOOGL.CloseTSLA.Close
2019-11-25 0.014083996 0.031063976 0.008022279 -0.009351804
2019-12-02 -0.002930276 -0.027701396 0.026708844 0.017872944
2019-12-09 0.006000325 -0.001645567 0.003621939 0.048834101
```

Statistics:

```
$mean
FB.CloseAMZN.CloseGOOGL.CloseTSLA.Close
0.0065254636 0.0018067212 0.0047195935 -0.0006965918

$Cov
FB.CloseAMZN.CloseGOOGL.CloseTSLA.Close
FB.Close 0.0015049227 0.0008843987 5.196284e-04 5.240888e-04
AMZN.Close 0.0008843987 0.0012159249 5.348228e-04 4.243135e-04
GOOGL.Close 0.0005196284 0.0005348228 8.658192e-04 -4.177369e-05
TSLA.Close 0.0005240888 0.0004243135 -4.177369e-05 5.090559e-03

$estimator
[1] "covEstimator"

$mu
FB.CloseAMZN.CloseGOOGL.CloseTSLA.Close
0.0065254636 0.0018067212 0.0047195935 -0.0006965918

$Sigma
FB.CloseAMZN.CloseGOOGL.CloseTSLA.Close
FB.Close 0.0015049227 0.0008843987 5.196284e-04 5.240888e-04
AMZN.Close 0.0008843987 0.0012159249 5.348228e-04 4.243135e-04
GOOGL.Close 0.0005196284 0.0005348228 8.658192e-04 -4.177369e-05
TSLA.Close 0.0005240888 0.0004243135 -4.177369e-05 5.090559e-03
```

After we have computed returns of stocks in consideration, we have plotted the histograms of return distributions for each individual stock. It represents good visualization about symmetry/asymmetry of corresponding return distribution.

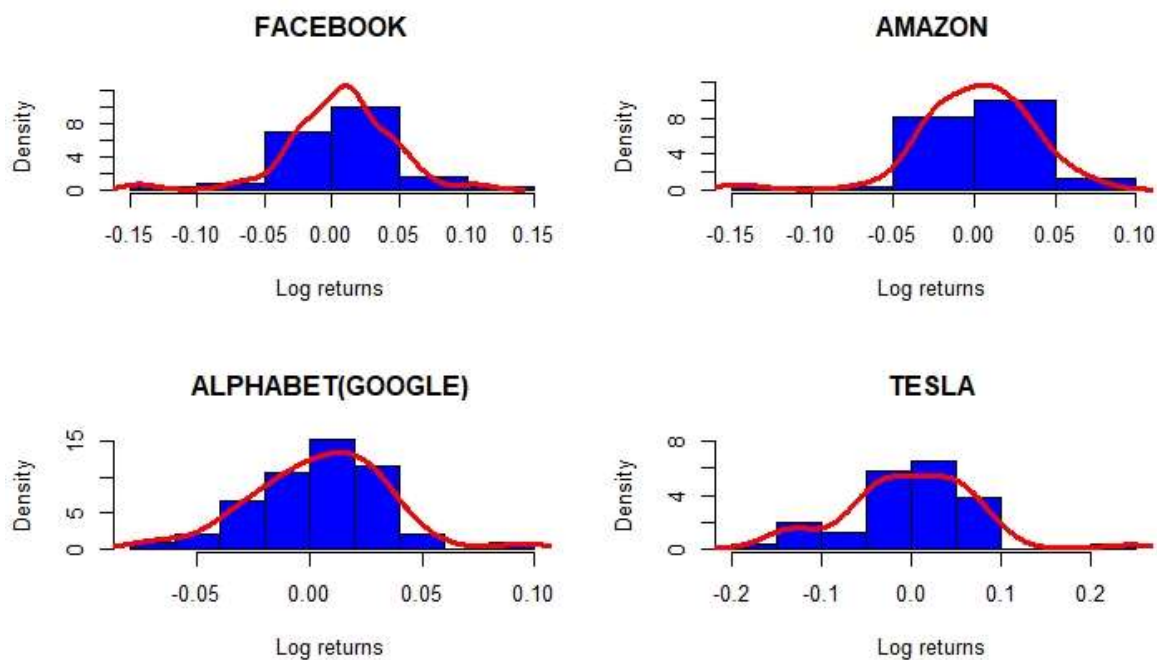


Figure 5: Histogram of asset returns

Source: Authors calculation (R-studio)

Histogram can be created using the `hist()` function in R programming language. The density function, represented by the histogram of returns, describes the relative likelihood for this random variable to take on a given value. It also acts as a tool to identify the skewness.



Skewness is used as an additional risk measure when the data shows asymmetrical distribution, which is the case with selected stocks in this analysis. Skewness can either be negative or positive. Positive skewness shows a return distribution where frequent small losses and a few extreme gains are common while negative skewness shows frequent small gains and a few extreme losses. If a stock's return follows a normal distribution pattern, then skewness will be 0. Figure 5 shows that the majority of returns of Tesla and Alphabet.inc are positive because they have longer or fatter tail on the right side of the distribution while the majority of the returns of Facebook and Amazon are negative because they have longer or fatter tail on the left side. The shape of the tails of the distribution is measured by the kurtosis of the distribution. Assuming rationality, investors should prefer low excess kurtosis because that indicates more predictable returns. Kurtosis greater than 3 indicates departure from normal distribution in most return series, or the fat-tails problem what is evident at Amazon followed by Facebook.

Code for the figure 3 in R is as follows

```
>hist(portfolioReturns$FB.Close, probability = T,breaks = 7,col = "blue",main = "FACEBOOK",
xlab = "Log returns",ylim = c(0,13))
>lines(density(portfolioReturns$FB.Close),col=2,lwd=3)
>hist(portfolioReturns$AMZN.Close, probability = T,breaks = 7,col = "blue",main = "AMAZON",
xlab = "Log returns",ylim = c(0,12))
>lines(density(portfolioReturns$AMZN.Close),col=2,lwd=3)
>hist(portfolioReturns$GOOGL.Close, probability = T,breaks = 7,col = "blue",main =
"ALPHABET(GOOGLE)", xlab = "Log returns",ylim = c(0,15))
>lines(density(portfolioReturns$GOOGL.Close),col=2,lwd=3)
>hist(portfolioReturns$TSLA.Close, probability = T,breaks = 7,col = "blue",main = "TESLA",
xlab = "Log returns",ylim = c(0,8))
>lines(density(portfolioReturns$TSLA.Close),col=2,lwd=3)
```

## 6.2 Mean-Variance in R

Computing portfolio components

The mean for each stock is calculated in R by applying function `colMeans()` on already calculated returns and will provide expected returns for chosen stocks.

```
>portfolioReturns

>mu.vec<-colMeans(portfolioReturns)
>names(mu.vec)<- asset.names
>asset.names<- c("FB", "AMZN", "GOOGL", "TSLA")

>mu.vec
      FB      AMZN      GOOGL      TSLA
0.0065254636 0.0018067212 0.0047195935 -0.0006965918
```

Covariances are often presented in the covariance matrix. From continuously compounded returns covariance matrix and correlation matrix are calculated.

There are several different ways to formulate covariance matrix in R. It can be done simply by using functions `cov()` and `cor()`.

```
>cov(portfolioReturns)
>cor(portfolioReturns)

>cov(portfolioReturns)
      FB      AMZN      GOOGL      TSLA
FB    0.0015049227 0.0008843987 5.196284e-04 5.240888e-04
AMZN  0.0008843987 0.0012159249 5.348228e-04 4.243135e-04
GOOGL 0.0005196284 0.0005348228 8.658192e-04 -4.177369e-05
TSLA  0.0005240888 0.0004243135 -4.177369e-05 5.090559e-03
>cor(portfolioReturns)
      FB      AMZN      GOOGL      TSLA
FB    1.00000000 0.6537891 0.45522085 0.18934988
AMZN  0.6537891 1.0000000 0.52124622 0.17054963
GOOGL 0.4552208 0.5212462 1.00000000 -0.01989787
TSLA  0.1893499 0.1705496 -0.01989787 1.00000000
```

It is also possible to obtain it in the following way. First, matrix M is constructed out of 4 vectors:

```
> FB <- c((portfolioReturns)[,1])
> AMZN <- c((portfolioReturns)[,2])
> GOOGL <- c((portfolioReturns)[,3])
> TSLA <- c((portfolioReturns)[,4])
> M <- cbind(FB,AMZN,GOOGL,TSLA)
```

Then a matrix of means (M\_mean) is created:

```
> k <- ncol(M) #numberofvariables
> n <- nrow(M) #numberofsubjects
M_mean<- matrix(data=1, nrow=n) %%% cbind(mean(FB),mean(AMZN),mean(GOOGL),mean(TSLA))
```

After that, an excess matrix (E) is created by subtracting the matrix of means (M\_mean) from the data matrix (M):

```
> E <- M - M_mean
```

Finally we get the covariance matrix sigma.mat by multiplying the transpose of the excess matrix (E) with the excess matrix E and the inverse of the number of time periods minus 1.

```
> C <- (n-1)^-1* t(E) %%% E
>sigma.mat<-C
>sigma.mat
      FB      AMZN      GOOGL      TSLA
FB    0.0015049227 0.0008843987 5.196284e-04 5.240888e-04
AMZN  0.0008843987 0.0012159249 5.348228e-04 4.243135e-04
GOOGL 0.0005196284 0.0005348228 8.658192e-04 -4.177369e-05
TSLA  0.0005240888 0.0004243135 -4.177369e-05 5.090559e-03
```

As we have mentioned before, covariance is a statistical descriptive measure that indicates relationship between returns of two securities. Covariance is negative if the return on one asset is above its expected value while the return on the other asset is below its expected value. Sigma.mat matrix reveals that only TSLA and GOOGL exhibit a negative relationship.

If TSLA price goes up, it is likely that GOOGL price will go down. Numbers on the diagonal of the matrix are variances of the portfolio stocks. These variances are actually covariances of the same stocks and represent risk of portfolio stocks. If variance is 0, there is no dispersion of returns or risk. Variance greater than 0 shows dispersion of outcome, the greater the number the greater the dispersion. According to the covariance matrix, the riskiest asset is TSLA because it has the highest variance. While covariance is meaningful because it influences portfolio risk, portfolio correlation coefficient is more useful because it standardizes covariance (Gibson, 2004).

R function for correlation is:

```
>cor(M)
```

	FB	AMZN	GOOGL	TSLA
FB	1.0000000	0.6537891	0.45522085	0.18934988
AMZN	0.6537891	1.0000000	0.52124622	0.17054963
GOOGL	0.4552208	0.5212462	1.0000000	-0.01989787
TSLA	0.1893499	0.1705496	-0.01989787	1.0000000

Correlation is a measure of strength and direction of linear relationship. It is a number between -1 and +1. A correlation of 0 indicates no linear relation between the two variables. Combining securities with zero correlation can reduce the risk of the portfolio. Correlation matrix shows that all portfolio stocks, except TSLA and GOOGL, have positive correlation indicating that positively correlated stocks move in the same direction. From modern portfolio theory standpoint, high correlation between stocks is not a good indicator. If stocks in the portfolio are highly correlated then when one of the stocks takes a downturn, probably entire portfolio will take downturn. Thus, for risk-averse investor it is important to have assets whose correlation is as low as possible. Stocks in the selected portfolio have correlation below 0.70 which is a good starting point. The highest correlation in the selected portfolio is between AMZN and FB (0.65) whereas the lowest correlation, and even negative, is between TSLA and GOOGL (-0.02). Negatively correlated assets are called hedging assets and they represent very effective way of reducing portfolio risk.

Figure 6 displays correlation map that brings better visualization of the strength of relationship between selected stocks.

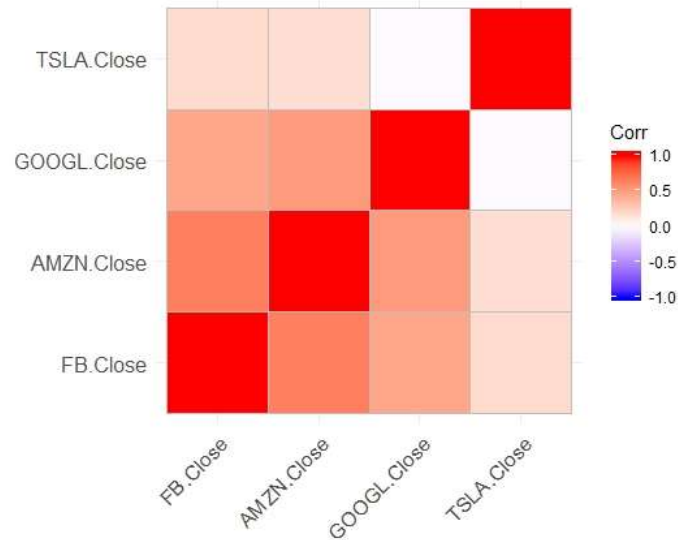


Figure 6:Correlation Map

Source: Authors creation (R –studio)

### 6.3 Finding the Envelope of Financial Assets

The first case considers solving the problem of minimizing risk with some target expected return if short sales are allowed. It will be solved as a constrained optimization problem by using the Lagrange multiplier method. This model does not have constraints on assets' weights described by vector  $\mathbf{x}$ , that is it permits short sales. In our example we present the use of matrix algebra which simplifies many of the computations. Results are implemented in RMetrics. Let  $x_i$  denote the share of wealth invested in stock  $i$ ,  $i=1,\dots,4$ , and assume that all wealth is invested in four assets considered so that

$$x_{FB} + x_{AMZ} + x_{GOOGL} + x_{TSLA} = 1.$$

The portfolio return,  $R_{px}$  is the random variable

$$R_{px} = x_{FB}R_{FB} + x_{AMZ}R_{AMZ} + x_{GOOGL}R_{GOOGL} + x_{TSLA}R_{TSLA}. \quad (10)$$

The expected return on the portfolio is equal to

$$R_{px} = \mu_{px} = x_{FB}\mu_{FB} + x_{AMZ}\mu_{AMZ} + x_{GOOGL}\mu_{GOOGL} + x_{TSLA}\mu_{TSLA}. \quad (11)$$

The variance of the portfolio return is equal to

$$\begin{aligned}
\sigma_{px}^2 = & x_{FB}^2 \sigma_{FB}^2 + x_{AMZ}^2 \sigma_{AMZ}^2 + x_{GOOGL}^2 \sigma_{GOOGL}^2 + x_{TSLA}^2 \sigma_{TSLA}^2 + 2x_{FB}x_{AMZ}\sigma_{FBAMZ} \\
& + 2x_{FB}x_{GOOGL}\sigma_{FBGOOGL} + 2x_{FB}x_{TSLA}\sigma_{FBTSLA} + 2x_{AMZ}x_{GOOGL}\sigma_{AMZGOOGL} \\
& + 2x_{AMZ}x_{TSLA}\sigma_{AMZTSLA} + 2x_{GOOGL}x_{TSLA}\sigma_{GOOGLTSLA}
\end{aligned}
\tag{12}$$

The variance of the portfolio return depends on four variance terms and 12 covariance terms. Even with four assets, the algebra representing the portfolio characteristics in terms of expected return and variance is very complicated. With matrix notation we can simplify calculation.

Return data using matrix notation from four selected stocks can be represented as

$$\mu = \begin{pmatrix} \mu_{FB} \\ \mu_{AMZ} \\ \mu_{GOOGL} \\ \mu_{TSLA} \end{pmatrix} = \begin{pmatrix} 0.00652 \\ 0.00181 \\ 0.00472 \\ -0.00069 \end{pmatrix}
\tag{13}$$

4 × 4 covariance matrix of returns is equal to

$$\Sigma = \begin{pmatrix} 0.001504 & 0.000884 & 0.000526 & 0.000524 \\ 0.000884 & 0.001216 & 0.000535 & 0.000424 \\ 0.00052 & 0.000535 & 0.000866 & -4.18E05 \\ 0.000524 & 0.000424 & -4.18E05 & 0.005091 \end{pmatrix}.
\tag{14}$$

Covariance matrix is symmetric since  $cov(R_{FB}, R_{AMZ}) = cov(R_{AMZ}, R_{FB})$ .

## 6.4 Portfolio Computations in R

First we construct an equally weighted portfolio

$x_{FB} = x_{AMZ} = x_{GOOGL} = x_{TSLA} = 1/4$  that has return  $R_{px} = \mathbf{x}^T \mathbf{R}$

```

>x.vec = rep(1,4)/4
>names(x.vec) <-asset.names
>mu.p.x<- crossprod(x.vec,mu.vec)
> sig2.p.x <- t(x.vec)%*%sigma.mat%*%x.vec
>sig.p.x<- sqrt(sig2.p.x)
>mu.p.x
      [,1]
[1,] 0.003088797
>sig.p.x
      [,1]

```

```
[1,] 0.02996684
```

Next, we choose another portfolio with weight vector

$\mathbf{y}=(y_{FB}, y_{AMZ}, y_{GOOGL}, y_{TSLA})=(0.25, 0.15, 0.5, 0.10)$  and return  $R_{py} = \mathbf{y}^T \mathbf{R}$

Covariance between  $R_{py}$  and  $R_{px}$  is

```
>y.vec = c(0.25, 0.15, 0.5, 0.10)
>names(x.vec) <- asset.names
>sig.xy<- t(x.vec)%*%sigma.mat%*%y.vec
>sig.xy
      [,1]
[1,] 0.0007140364
```

## Global Minimum Variance Portfolio

Global minimum variance portfolio  $\mathbf{gm}=(gm_{FB}, gm_{AMZN}, gm_{GOOGL}, gm_{TSLA})$  solves the constrained minimization problem

$$\text{Min } \sigma_{pgm}^2 = \mathbf{gm}^T \Sigma \mathbf{gm} \text{ s.t. } \mathbf{gm}^T \mathbf{1} = 1$$

The Lagrangian function for this problem is as follows:

$$\begin{aligned} L(gm_{FB}, gm_{AMZN}, gm_{GOOGL}, gm_{TSLA}, \lambda) = & gm_{FB}^2 \sigma_{FB}^2 + gm_{AMZ}^2 \sigma_{AMZ}^2 + \\ & gm_{GOOGL}^2 \sigma_{GOOGL}^2 + gm_{TSLA}^2 \sigma_{TSLA}^2 + 2gm_{FB}gm_{AMZ}\sigma_{FBAMZ} + \\ & 2gm_{FB}gm_{GOOGL}\sigma_{FBGOOGL} + 2gm_{FB}gm_{TSLA}\sigma_{FBTSLA} + 2gm_{AMZ}gm_{GOOGL}\sigma_{AMZGOOGL} + \\ & 2gm_{AMZ}gm_{TSLA}\sigma_{AMZTSLA} + 2gm_{GOOGL}gm_{TSLA}\sigma_{GOOGLTSLA} + \lambda(gm_{FB} + gm_{AMZN} + \\ & gm_{GOOGL} + gm_{TSLA} - 1) \end{aligned} \quad (15)$$

The first order conditions for minimum are

$$\begin{aligned} 0 = \frac{\partial L}{\partial gm_{FB}} = & 2gm_{FB}\sigma_{FB}^2 + 2gm_{AMZ}\sigma_{FBAMZ} + 2gm_{GOOGL}\sigma_{FBGOOGL} + 2gm_{TSLA}\sigma_{FBTSLA} \\ & + \lambda \end{aligned}$$

$$\begin{aligned}
0 &= \frac{\partial L}{\partial gm_{AMZ}} = 2gm_{AMZ}\sigma^2_{AMZ} + 2gm_{FB}\sigma_{FBAMZ} + 2gm_{GOOGL}\sigma_{AMZGOOGL} \\
&\quad + 2gm_{TSLA}\sigma_{AMZTSLA} + \lambda \\
0 &= \frac{\partial L}{\partial gm_{GOOGL}} = 2gm_{GOOGL}\sigma^2_{GOOGL} + 2gm_{FB}\sigma_{FBGOOGL} + 2gm_{AMZ}\sigma_{AMZGOOGL} \\
&\quad + 2gm_{TSLA}\sigma_{GOOGLTSLA} + \lambda \\
0 &= \frac{\partial L}{\partial gm_{TSLA}} = 2gm_{TSLA}\sigma^2_{TSLA} + 2gm_{FB}\sigma_{FBTSLA} + 2gm_{AMZ}\sigma_{AMZTSLA} \\
&\quad + 2gm_{GOOGL}\sigma_{GOOGLTSLA} + \lambda \\
0 &= \frac{\partial L}{\partial \lambda} = gm_{FB} + gm_{AMZN} + gm_{GOOGL} + gm_{TSLA} - 1
\end{aligned} \tag{16}$$

The solutions of the system of five linear equations obtained from the first order conditions (24) are the weights of the assets in the global minimum variance portfolio. In a matrix form the equations of the first order conditions can be written as

$$\begin{pmatrix}
2\sigma_{FB}^2 & 2\sigma_{FBAMZN} & 2\sigma_{FBGOOGL} & 2\sigma_{FBTSLA} & 1 \\
2\sigma_{FBAMZN} & 2\sigma_{AMZN}^2 & 2\sigma_{AMZNGOOGL} & 2\sigma_{AMZNTSLA} & 1 \\
2\sigma_{FBGOOGL} & 2\sigma_{AMZNGOOGL} & 2\sigma_{GOOGL}^2 & 2\sigma_{GOOGLTSLA} & 1 \\
2\sigma_{FBTSLA} & 2\sigma_{AMZNTSLA} & 2\sigma_{GOOGLTSLA} & 2\sigma_{TSLA}^2 & 1 \\
1 & 1 & 1 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
gm_{FB} \\
gm_{AMZN} \\
gm_{GOOGL} \\
gm_{TSLA} \\
\lambda
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
1
\end{pmatrix}$$

In shorter notation, it can be written as

$$\begin{pmatrix} 2\Sigma & \mathbf{1} \\ \mathbf{1}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{gm} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}$$

$$\text{or } \mathbf{A}_{gm} \mathbf{z}_{gm} = \mathbf{b}$$

(17)

Where

$$\mathbf{A}_{gm} = \begin{pmatrix} 2\Sigma & \mathbf{1} \\ \mathbf{1}^T & 0 \end{pmatrix}, \mathbf{z}_{gm} = \begin{pmatrix} \mathbf{gm} \\ \lambda \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}$$

The solution is

$$\mathbf{z}_{gm} = \mathbf{A}_{gm}^{-1} \mathbf{b}. \quad (18)$$

The first four elements of  $\mathbf{z}_{gm}$  are the weights for the global minimum variance portfolio with expected return  $\mu_{pgm} = \mathbf{gm}^T \boldsymbol{\mu}$  and variance  $\sigma_{pgm}^2 = \mathbf{gm}^T \Sigma \mathbf{gm}$ .

Using the data for the expected returns and variances from selected stocks and equation (24) we compute the weight of the assets in the global minimum variance portfolio. In R we do it by running the following set of commands:

```
>top.mat<- cbind(2*sigma.mat, rep(1, 4))
>bot.vec<- c(rep(1, 4), 0)
>Am.mat<- rbind(top.mat, bot.vec)
>b.vec<- c(rep(0, 4), 1)
>z.m.mat<- solve(Am.mat)%*%b.vec
>m.vec<- z.m.mat[1:4,1]
>top.mat<- cbind(2*sigma.mat, rep(1, 4))
>bot.vec<- c(rep(1, 4), 0)
>Am.mat<- rbind(top.mat, bot.vec)
>b.vec<- c(rep(0, 4), 1)
>z.m.mat<- solve(Am.mat)%*%b.vec
>gm.vec<- z.m.mat[1:4,1]
>gm.vec
```

FB	AMZN	GOOGL	TSLA
0.09065791	0.17538412	0.62052740	0.11343058

The global minimum variance portfolio has weights

$$gm_{FB} = 0.0906, gm_{AMZN} = 0.1754, gm_{GOOGL} = 0.6205, gm_{TSLA} = 0.1134$$

The expected return on this portfolio  $\mu_{pgm} = \mathbf{gm}^T \boldsymbol{\mu}$  is obtained as

```
>mu.gmin<- as.numeric(crossprod(gm.vec, mu.vec))
>mu.gmin
[1] 0.003758077
```

The portfolio variance  $\sigma_{pgm}^2 = \mathbf{gm}^T \Sigma \mathbf{gm}$  and standard deviation  $\sigma_{pgm}$  are

```
> sig2.gmin <- as.numeric(t(gm.vec)%*%sigma.mat%*%gm.vec)
>sig.gmin<- sqrt(sig2.gmin)
> sig2.gmin
[1] 0.000673434
>
>sig.gmin
[1] 0.02595061
```

## Efficient portfolio



As it has already been stated, efficient portfolios lie in the upper part of the minimum variance set that is the part above the global minimum variance portfolio. According to Markowitz framework, we assume that investors wish to find portfolios that have the best expected return-risk trade-off. In this analysis, investors seek to find portfolios that minimize portfolio's risk (measured by portfolio variance) subject to a target expected return level.

$$\text{Min } \sigma_{px}^2 = \mathbf{x}^T \Sigma \mathbf{x} \text{ s.t. } \mathbf{x}^T \mathbf{1} = 1 \text{ and } \mu_p = \mathbf{x}^T \boldsymbol{\mu} = \mu_{p0} \quad (19)$$

The efficient portfolio frontier is a graph of values of expected returns and variances for the set of efficient portfolios created by solving equation (25) for all possible target expected return levels which are greater than the expected return of the global minimum variance portfolio.

The corresponding Lagrangian function is equal to

$$L(x, \lambda_1, \lambda_2) = \mathbf{x}^T \Sigma \mathbf{x} + \lambda_1 (\mathbf{x}^T \boldsymbol{\mu} - \mu_{p0}) + \lambda_2 (\mathbf{x}^T \mathbf{1} - 1).$$

Now we have two Lagrange multipliers because we have two constraints.

From the first order conditions we get

$$\frac{\partial L}{\partial x} = 2\Sigma \mathbf{x} + \lambda_1 \boldsymbol{\mu} + \lambda_2 \mathbf{1} = 0$$

$$\frac{\partial L}{\partial \lambda_1} = \mathbf{x}^T \boldsymbol{\mu} - \mu_{p0} = 0$$

$$\frac{\partial L}{\partial \lambda_2} = \mathbf{x}^T \mathbf{1} - 1 = 0$$

In the matrix form, this can be represented as

$$\begin{pmatrix} 2\Sigma & \boldsymbol{\mu} & \mathbf{1} \\ \boldsymbol{\mu}^T & 0 & 0 \\ \mathbf{1}^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mu_{p0} \\ 1 \end{pmatrix}$$

Or

$$\mathbf{A}\mathbf{z}_x = \mathbf{b}_0 \quad (20)$$

Where

$$\mathbf{A} = \begin{pmatrix} 2\mathbf{\Sigma} & \boldsymbol{\mu} & \mathbf{1} \\ \boldsymbol{\mu}^T & 0 & 0 \\ \mathbf{1}^T & 0 & 0 \end{pmatrix}, \mathbf{z}_x = \begin{pmatrix} \mathbf{x} \\ \lambda_1 \\ \lambda_2 \end{pmatrix} \text{ and } \mathbf{b}_0 = \begin{pmatrix} \mathbf{0} \\ \mu_{p0} \\ 1 \end{pmatrix}$$

Solution for  $\mathbf{z}_x$  is equal to

$$\mathbf{z}_x = \mathbf{A}^{-1}\mathbf{b}_0 \quad (21)$$

The first three elements of  $\mathbf{z}_x$  are the weights for the minimum variance portfolio with expected return  $\mu_{px} = \mu_{p0}$ . In order to obtain efficient portfolios, we solve constrained minimization problem (26) by using equation (28) for expected returns  $\mu_{p0}$  greater or equal to the expected return of the global minimum variance portfolio  $\mu_{pgm} = 0.00375$ . In the following examples we will see that diversification yields benefit to investors. We will calculate two portfolios that have the same expected returns as the single-assets portfolios (Facebook and Google) but due to diversification they will have lower variance or risk.

First we calculate in R a mean-variance portfolio  $\mathbf{x} = (x_{FB}, x_{AMZN}, x_{GOOGL}, x_{TSLA})$  with the same expected return as Facebook;  $\mu_{FB} = 0.0065$ . Hence, target expected return will be  $\mu_{p0} = \mu_{FB} = 0.0065$ .

```
>top.mat<- cbind(2*sigma.mat, mu.vec, rep(1, 4))
>mid.vec<- c(mu.vec, 0, 0)
>bot.vec<- c(rep(1, 4), 0, 0)
>A.mat<- rbind(top.mat, mid.vec, bot.vec)
>bfb.vec = c(rep(0, 4), mu.vec["FB"], 1)
>z.mat<- solve(A.mat)%*%bfb.vec
>x.vec<- z.mat[1:4,]
```

The efficient portfolio with the same expected return as Facebook has portfolio weights  $\mathbf{x} = (x_{FB}, x_{AMZN}, x_{GOOGL}, x_{TSLA})$ .

(22)

```
>x.vec
      FB      AMZN      GOOGL      TSLA
0.54294471 -0.33764563  0.76550396  0.02919696
```

The expected return on this portfolio is equal to the target return  $\mu_{FB} = 0.0065$

```
>mu.px = as.numeric(crossprod(x.vec, mu.vec))
>mu.px
```

```
[1] 0.006525464
```

The portfolio variance  $\sigma_{px}^2 = \mathbf{x}^T \Sigma \mathbf{x}$  and standard deviation  $\sigma_{px}$  are equal to

```
> sig2.px = as.numeric(t(x.vec)%*%sigma.mat%*%x.vec)
> sig.px = sqrt(sig2.px)
> sig2.px
[1] 0.0009315558
> sig.px
[1] 0.0305214
```

It can be noticed that the calculated variance is smaller than the variance for Facebook alone with the same expected return.

Next, by using equation (25) we calculate the minimum variance portfolio  $\mathbf{y} = (Y_{FB}, Y_{AMZN}, Y_{GOOGL}, Y_{TSLA})$  with the same expected return as Google;  $\mu_{GOOGL} = 0.00472$ . This portfolio is efficient because target return of portfolio  $\mathbf{y}$  is greater than expected return of global minimum variance portfolio  $\mu_{GOOGL}$  is  $> \mu_{pgm}$ .

Weights for  $\mathbf{y} = (Y_{FB}, Y_{AMZN}, Y_{GOOGL}, Y_{TSLA})$  are

(23)

```
> bgoogl.vec = c(rep(0, 4), mu.vec["GOOGL"], 1)
> z.mat = solve(A.mat)%*%bgoogl.vec
> y.vec = z.mat[1:4,]
> y.vec
      FB      AMZN      GOOGL      TSLA
0.247802964 -0.002865809 0.670898867 0.084163977
```

The portfolio  $\mathbf{y}$ 's expected return and standard deviation are:

```
> mu.py = as.numeric(crossprod(y.vec, mu.vec))
> sig2.py = as.numeric(t(y.vec)%*%sigma.mat%*%y.vec)
> sig.py = sqrt(sig2.py)
> mu.py
[1] 0.004719594
> sig2.py
[1] 0.000704594
> sig.py
[1] 0.02654419
```

The covariance between the portfolio returns  $R_{px}$  and  $R_{py}$  is as follows:

```
> sigma.xy = as.numeric(t(x.vec)%*%sigma.mat%*%y.vec)
> rho.xy = sigma.xy/(sig.px*sig.py)
> sigma.xy
[1] 0.0007631173
> rho.xy
[1] 0.9419272
```

## Computing the efficient frontier

The linear combination of two mean-variance efficient portfolios gives a third mean-variance efficient portfolio, provided the new portfolio expected return is not smaller than that of the minimum-variance portfolio (this is true in particular for convex combinations) <sup>28</sup>

In following example, the linear combination of two efficient portfolios will yield two portfolios, one located on the efficient frontier and the other located on the inefficient part of the envelope set.

Let  $\mathbf{y} = (y_{FB}, y_{AMZN}, y_{GOOGL}, y_{TSLA})$  and  $\mathbf{x} = (x_{FB}, x_{AMZN}, x_{GOOGL}, x_{TSLA})$  be two minimum variance efficient portfolios with different target expected returns.

Portfolio  $\mathbf{x}$  solves

$$\text{Min} \sigma_{px}^2 = \mathbf{x}^T \Sigma \mathbf{x} \text{ s.t. } \mathbf{x}^T \mathbf{1} = 1 \text{ and } \mathbf{x}^T \boldsymbol{\mu} = \mu_{p0}$$

and portfolio  $\mathbf{y}$  solves

$$\text{Min} \sigma_{py}^2 = \mathbf{y}^T \Sigma \mathbf{y} \text{ s.t. } \mathbf{y}^T \mathbf{1} = 1 \text{ and } \mathbf{y}^T \boldsymbol{\mu} = \mu_{p1}.$$

Let  $\alpha$  be any constant. We define portfolio  $\mathbf{z}$  as a linear combination of portfolios  $\mathbf{x}$  and  $\mathbf{y}$ , which is:

$$\mathbf{z} = \alpha * \mathbf{x} + (1 - \alpha) * \mathbf{y} \quad (24)$$

Hence portfolio  $\mathbf{z}$  is a minimum variance portfolio with expected return and variance equal to

$$\mu_{pz} = \mathbf{z}^T \boldsymbol{\mu} = \alpha * \mu_{px} + (1 - \alpha) * \mu_{py} \quad (25)$$

$$\sigma_{pz}^2 = \mathbf{z}^T \Sigma \mathbf{z} = \alpha^2 \sigma_{px}^2 + (1 - \alpha)^2 \sigma_{py}^2 + 2 \alpha (1 - \alpha) \sigma_{xy} \quad (26)$$

Where

$$\sigma_{px}^2 = \mathbf{x}^T \Sigma \mathbf{x}$$

$$\sigma_{py}^2 = \mathbf{y}^T \Sigma \mathbf{y}$$

$$\sigma_{xy} = \mathbf{x}^T \Sigma \mathbf{y}$$

In R, for any number  $\alpha$  (here we use  $\alpha = 0.5$ ) portfolio  $\mathbf{z}$  can be computed as follows:

```
> a = 0.5
> z.vec <- a * x.vec + (1-a) * y.vec
> z.vec
```

---

<sup>28</sup>Luciano, Elisa & Dumas, Bernard. (2017). The Economics of Continuous-time Finance, MIT Press.

	FB	AMZN	GOOGL	TSLA
	0.39537384	-0.17025572	0.71820142	0.05668047

Note that portfolio  $\mathbf{z}$  is not necessarily efficient.

The expected return, variance and standard deviation of portfolio  $\mathbf{z}$  are

```
>mu.pz<- as.numeric(crossprod(z.vec, mu.vec))
> sig2.pz <- as.numeric(t(z.vec)%*%sigma.mat%*%z.vec)
>sig.pz<- sqrt(sig2.pz)

>mu.pz
[1] 0.005622529

> sig2.pz
[1] 0.0007905961

>sig.pz
[1] 0.02811754
```

Next we compute portfolio with the expected return equal to the expected return on Amazon using two already calculated efficient portfolios (equations (28) and (29))

$$\mu_{pz} = \alpha\mu_{px} + (1 - \alpha)\mu_{py} = \mu_{AMZN} = 0.001807$$

$$\text{Solving for constant } \alpha = \frac{\mu_{AMZN} - \mu_{py}}{\mu_{px} - \mu_{py}} = \frac{0.001807 - 0.004719}{0.006525 - 0.004719} = -1.612$$

In R, weights for portfolio  $\mathbf{z}_{amzn}$  are computed as follows:

```
>a.amzn<- (mu.vec["AMZN"] - mu.py)/(mu.px - mu.py)
>z.amzn<- a.amzn*x.vec + (1 - a.amzn)*y.vec

>z.amzn
      FB      AMZN      GOOGL      TSLA
-0.2282613  0.5371348  0.5183006  0.1728259
```

The expected return, variance and standard deviation on this portfolio are

```
>mu.pz.amzn = as.numeric(crossprod(z.amzn, mu.vec))
> sig2.pz.amzn = as.numeric(t(z.amzn)%*%sigma.mat%*%z.amzn)
>sig.pz.amzn = sqrt(sig2.pz.amzn)
>mu.pz.amzn

[1] 0.001806721
>
> sig2.pz.amzn
[1] 0.0008017729

>sig.pz.amzn
[1] 0.02831559
```

$\mathbf{z}_{AMZN}$  is inefficient portfolio because it's expected return is smaller than expected return on the global minimum variance portfolio.

## Plotting efficient frontier

In order to compute the efficient frontier, which is the part of envelope portfolios whose expected return is greater than the expected return on the global minimum variance portfolio,

first we compute the minimum variance set by using equation (25) and two efficient portfolios. For the first portfolio we use the global minimum variance and for the second we use the portfolio  $\mathbf{x}$  whose target expected return is equal to the highest expected return among single asset portfolios in consideration. It must be noted that a convex combination of two efficient portfolios is not necessarily efficient, but for sure it is on the envelope.

To compute and plot the efficient frontier in R we use:

- Global minimum variance portfolio:  $\sigma_{pgm}^2 = \mathbf{gm}^T \Sigma \mathbf{gm}$  and  $\mu_{pgm} = \mathbf{gm}^T \boldsymbol{\mu}$
- Some efficient portfolio  $\mathbf{x}$  with target expected return equal to the maximum expected return of the assets under consideration, in our case Facebook;  $\sigma_{px}^2 = \mathbf{x}^T \Sigma \mathbf{x}$  and  $\mu_{px} = \mathbf{x}^T \boldsymbol{\mu}$ ; or some other efficient portfolio
- Covariance of the two above calculated portfolios;  $\sigma_{gmx} = \mathbf{gm}^T \Sigma \mathbf{x}$

Once we have that, we have to:

- Set an initial grid of  $\alpha$  values and calculate frontier portfolios  $\mathbf{z}$  using equations (28), (29) and (30).
- Plot  $\mu_{pz}$  and  $\sigma_{pz}$  and adjust values of  $\alpha$  so that portfolios can be visible

```
> a = seq(from=1, to=-5, by=-0.15)
a = seq(from=-5, to=5, by=0.09)

>n.a = length(a)
>z.mat = matrix(0, n.a, 4)
>mu.z = rep(0, n.a)
> sig2.z = rep(0, n.a)
> sig.mx = t(gm.vec)%*%sigma.mat%*%x.vec
> for (i in 1:n.a) {
+   z.mat[i, ] = a[i]*gm.vec + (1-a[i])*x.vec
+   mu.z[i] = a[i]*mu.gmin + (1-a[i])*mu.px
+   sig2.z[i] = a[i]^2 * sig2.gmin + (1-a[i])^2 * sig2.px +
+     2*a[i]*(1-a[i])*sig.mx }
>
> plot(sqrt(sig2.z), mu.z, type="b", ylim=c(-0.0009, 0.008), xlim=c(0, 0.08 ),
+       pch=16, col="blue", ylab=expression(mu[p]),
+       xlab=expression(sigma[p]))
>
>text(sig.gmin, mu.gmin, labels="Global min", pos=2)
>text(sig.px, mu.px, labels="x=μ_fb", pos=2)
>text(sig.py, mu.py, labels="y=μ_googl", pos=2)
>text(sig.pz.amzn, mu.pz.amzn, labels="z_amzn", pos=2)
>text(sdFb, mu.vec["FB"], labels="FB", pos=4)
>text(sdgoogl, mu.vec["GOOGL"], labels="GOOGL", pos=4)
>text(sig.pz, mu.pz, labels="z", pos=4)
>text(sdamzn, mu.vec["AMZN"], labels="AMZN", pos=4)
>text(sdtsla, mu.vec["TSLA"], labels="T", pos=4)
>
>
>points(sig.pz.amzn, mu.pz.amzn, col="red")
>points(sig.px, mu.px, col="red")
>points(sig.py, mu.py, col="red")
>points(sig.gmin, mu.gmin, col="red")
>points(sdFb, mu.vec["FB"], col="red")
>points(sdgoogl, mu.vec["GOOGL"], col="red")
>points(sig.pz, mu.pz, col="red")
>points(sdamzn, mu.vec["AMZN"], col="red")
>points(sdtsla, mu.vec["TSLA"], col="red")
```

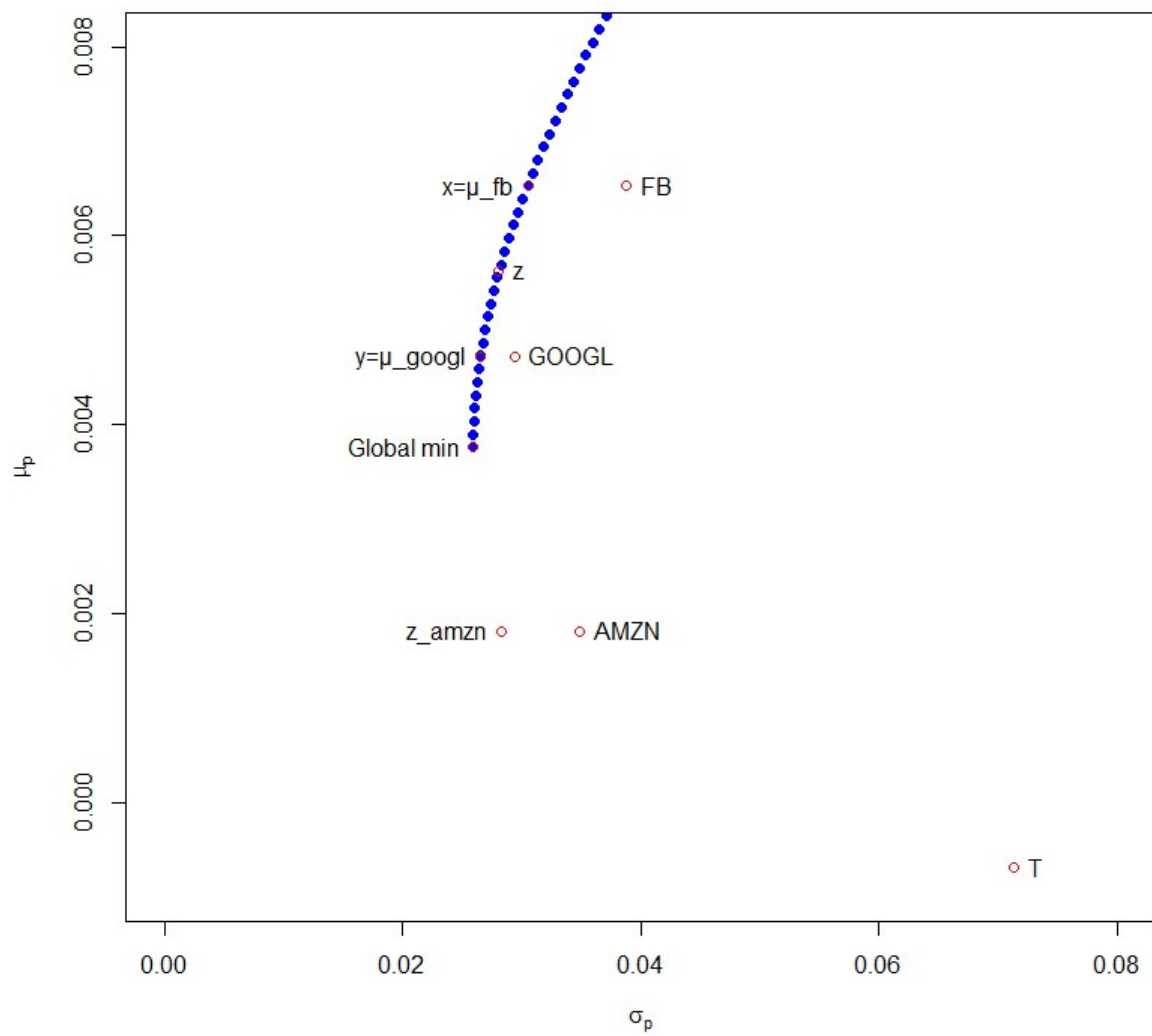


Figure 7:Efficient Frontier

Source: Authors creation (R-studio)

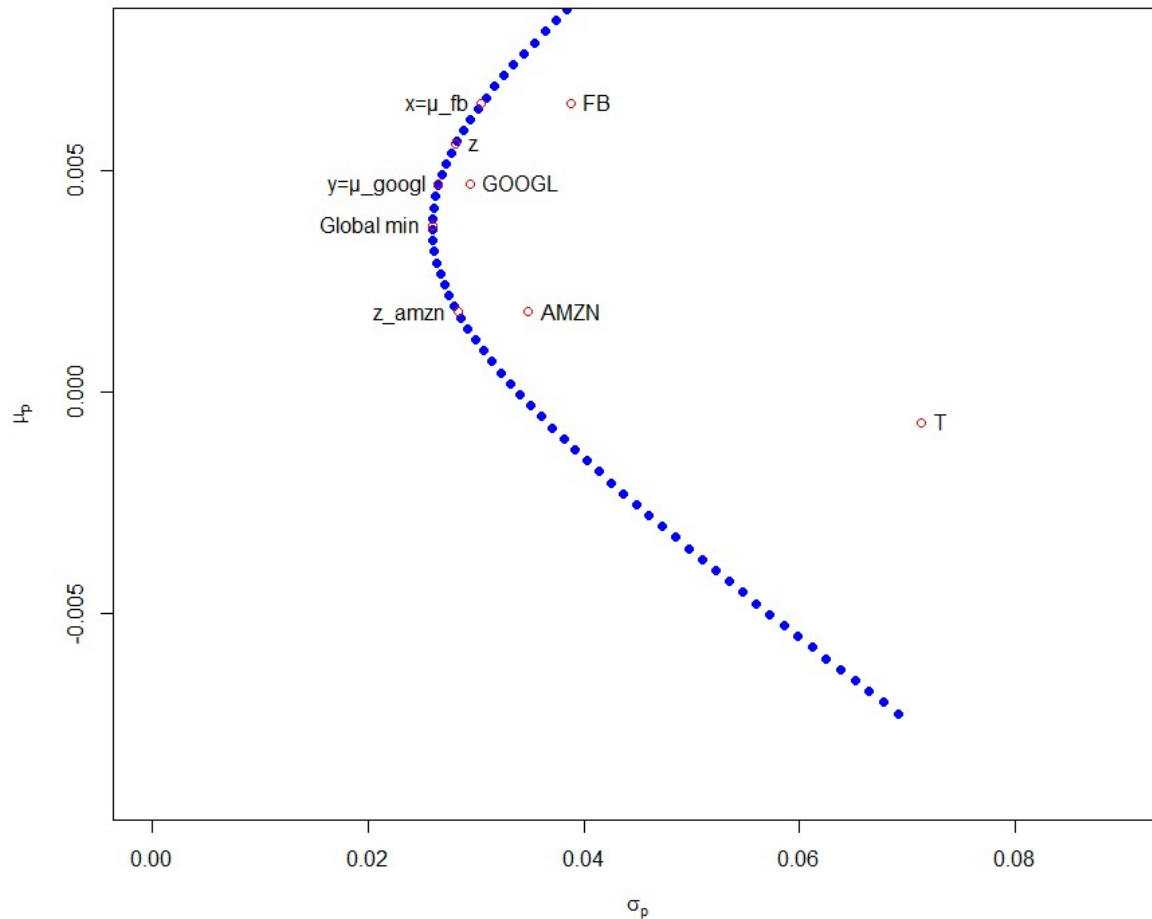


Figure 8:Envelope set

Source: Authors creation (R-studio)

Figure 7 shows the efficient frontier and figure 8 shows the envelope set of portfolios containing four selected stocks. As it has already been mentioned, global minimum variance portfolio is the border portfolio on the envelope set that separates inefficient part from the efficient frontier. Thus, it is the point where efficient frontier starts. In our graphs global minimum variance portfolio is dominated by other portfolios, such as portfolios  $y$ ,  $x$  and  $z$  that offer better expected return for the same level of risk. Figures 7 and 8 also show the means and standard deviations of the three stocks (AMZN, GOOGL and FB) separately. They are plotted off to the right of the envelope set in the feasible set. If we move directly to the left



from single asset portfolios, we will reach a portfolio on the frontier that has smaller risk without affecting the return (portfolio AMZN and `z_amzn`). However, if we move directly up from these individual portfolios, we will reach a portfolio that has greater return with the same level of risk. These single asset portfolios (AMZN, GOOGL and FB) are not efficient for an investor who can allocate her/his wealth across all four financial assets. From this example we can draw conclusions about the advantages of portfolio diversification, i.e. that we can improve the risk-return trade-off by increasing the number of financial assets in which we can invest.

## 6.4 Mean-Variance Portfolio Computations in R

Mean-variance (MV) portfolios in R are specified by time series data, portfolio specification object and constraints. Here we use the default settings for `portfolioSpec` which deals with MV portfolio and minimizes the risk using the `quadprog` solver. Feasible, efficient and global minimum risk portfolio will be calculated using functions `feasiblePortfolio`, `efficientPortfolio` and `minvariancePortfolio`

We consider the case of long-only constraints, that is the case when the weights of the assets are non-negative.

### Feasible portfolio

To compute a feasible portfolio we need to specify the portfolio weights. In this case we will define equal weights portfolio for our data set of returns. Then we can compute properties of the feasible portfolio EW using function `feasiblePortfolio()`.

```
>ewSpec<- portfolioSpec()
>nAssets<- ncol(portfolioReturns)
>setWeights(ewSpec) <- rep(1/nAssets, times = nAssets)
>ewPortfolio<- feasiblePortfolio(
+   data = portfolioReturns,
+   spec = ewSpec,
+   constraints = "LongOnly")
> print(ewPortfolio)
```

```
Title:
  MV FeasiblePortfolio
Estimator:      covEstimator
Solver:         solveRquadprog
Optimize:       minRisk
Constraints:     LongOnly
```

```
PortfolioWeights:
  FB  AMZN  GOOGL  TSLA
0.25 0.25  0.25  0.25
```

```

CovarianceRiskBudgets:
  FB    AMZN  GOOGL  TSLA
0.2389 0.2129 0.1307 0.4174

Target ReturnsandRisks:
meanCovCVarVaR
0.0031 0.0300 0.0721 0.0390

```

## Efficient portfolio

Efficient portfolio is a portfolio which has the lowest risk for some given level of return. Using the same target return as Google portfolio and EW portfolio, we compute efficient portfolios G and E using function `efficientPortfolio()`.

It can be noticed that portfolio G has the lower risk than portfolio Google for the same expected return. The same conclusion is reached for portfolios EW and E.

```

>mu.vec["GOOGL"]
      GOOGL
0.004719594

>sdgoogl<-sqrt(sum((GOOGL-mean(GOOGL))^2/(length(GOOGL)-1)))
>sdgoogl
[1] 0.02942481

```

```

>minriskSpec<- portfolioSpec()
>targetReturn<-0.0047
>setTargetReturn(minriskSpec) <- targetReturn
>G_Portfolio<- efficientPortfolio(
+   data = portfolioReturns,
+   spec = minriskSpec,
+   constraints = "LongOnly")
> print(G_Portfolio)

```

```

Title:
  MV EfficientPortfolio
Estimator:      covEstimator
Solver:         solveRquadprog
Optimize:       minRisk
Constraints:     LongOnly

```

```

PortfolioWeights:
  FB    AMZN  GOOGL  TSLA
0.2446 0.0008 0.6699 0.0848

```

```

CovarianceRiskBudgets:
  FB    AMZN  GOOGL  TSLA
0.2648 0.0007 0.6705 0.0641

```

```

Target ReturnsandRisks:
meanCovCVarVaR
0.0047 0.0265 0.0560 0.0358

```

```

>minriskSpec<- portfolioSpec()
>targetReturn<-0.0031
>setTargetReturn(minriskSpec) <- targetReturn
>E_Portfolio<- efficientPortfolio(
+   data = portfolioReturns,
+   spec = minriskSpec,
+   constraints = "LongOnly")
> print(E_Portfolio)

```

```

Title:
  MV EfficientPortfolio
Estimator:      covEstimator
Solver:         solveRquadprog
Optimize:       minRisk
Constraints:     LongOnly

```

```

PortfolioWeights:
  FB    AMZN  GOOGL  TSLA

```

```
0.0000 0.3028 0.5610 0.1362
CovarianceRiskBudgets:
  FB    AMZN    GOOGL    TSLA
0.0000 0.3192 0.5230 0.1578
Target ReturnsandRisks:
meanCovCVarVaR
0.0031 0.0262 0.0596 0.0360
```

## Global minimum varianceportfolio

Now we compute the global minimum varianceportfolio GM, which is the efficient portfolio with the lowest possible risk. We do this by using function `minvariancePortfolio()`.

```
>globminSpec<- portfolioSpec()
>globminPortfolio<- minvariancePortfolio(
+   data = portfolioReturns,
+   spec = globminSpec,
+   constraints = "LongOnly")
> print(globminPortfolio)
```

```
Title:
  MV Minimum VariancePortfolio
Estimator:      covEstimator
Solver:         solveRquadprog
Optimize:       minRisk
Constraints:     LongOnly
```

```
PortfolioWeights:
  FB    AMZN    GOOGL    TSLA
0.0907 0.1754 0.6205 0.1134
```

```
CovarianceRiskBudgets:
  FB    AMZN    GOOGL    TSLA
0.0907 0.1754 0.6205 0.1134
```

```
Target ReturnsandRisks:
```

```
meanCovCVarVaR
0.0038 0.0260 0.0572 0.0342
```

R output 1

## Mean-variance portfolio frontiers

Function `portfolioFrontier()` computes portfolios along the efficient frontier and minimum variance set. It is very simple to compute the efficient frontier for the default MV setting. There is only a few functions needed.

First we compute the efficient frontier for the selected stocks.

```
>Frontier<- portfolioFrontier(portfolioReturns)
> print(Frontier)
```

```
Title:
  MV PortfolioFrontier
Estimator:      covEstimator
Solver:         solveRquadprog
Optimize:       minRisk
Constraints:     LongOnly
PortfolioPoints: 5 of 49
```

```

PortfolioWeights:
      FB    AMZN    GOOGL    TSLA
1  0.0000  0.0589  0.0000  0.9411
13 0.0000  0.7504  0.0070  0.2427
25 0.0000  0.3294  0.5281  0.1425
37 0.2562  0.0000  0.6653  0.0785
49 1.0000  0.0000  0.0000  0.0000

CovarianceRiskBudgets:
      FB    AMZN    GOOGL    TSLA
1  0.0000  0.0061  0.0000  0.9939
13 0.0000  0.6682  0.0024  0.3294
25 0.0000  0.3517  0.4758  0.1725
37 0.2798  0.0000  0.6640  0.0562
49 1.0000  0.0000  0.0000  0.0000

Target ReturnsandRisks:
meanCovCVarVaR
1  -0.0005  0.0675  0.1349  0.1287
13 0.0012  0.0338  0.0812  0.0434
25 0.0030  0.0264  0.0608  0.0365
37 0.0048  0.0266  0.0563  0.0358
49 0.0065  0.0388  0.0944  0.0552

```

Figure 9 shows weights along the envelope set of a long-only constrained mean-variance portfolio. The upper axis labels the target risk and the lower the target return. The black vertical line separates the efficient frontier of the inefficient part of the envelope set. Hence, the risk axis increases to both sides from the vertical line.

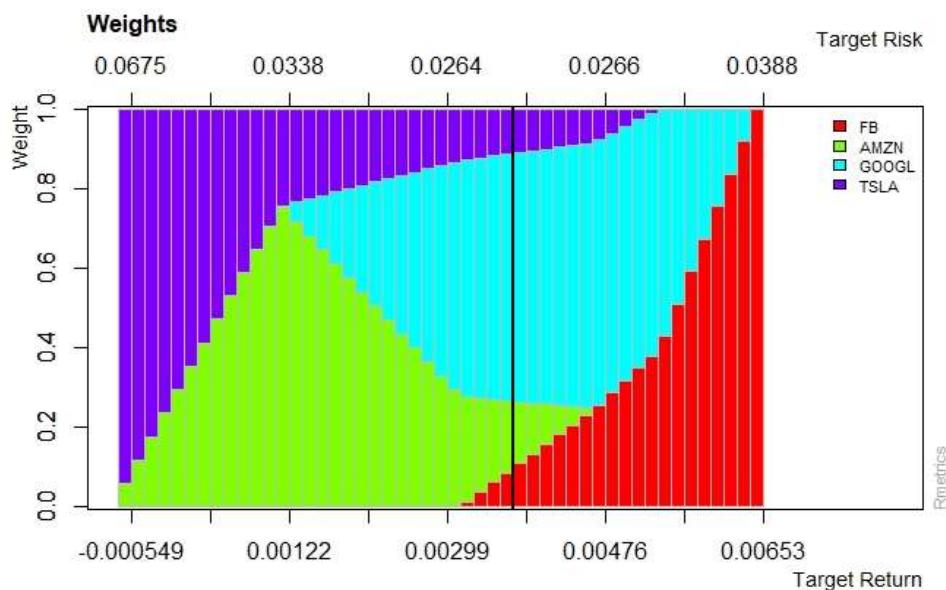


Figure 9:Weights along envelope set

Source: Authors creation (R-studio)

Now we can plot the efficient frontier using generic function `plot()` and assigning points and texts to corresponding portfolios using function `text()` and `points()`

```

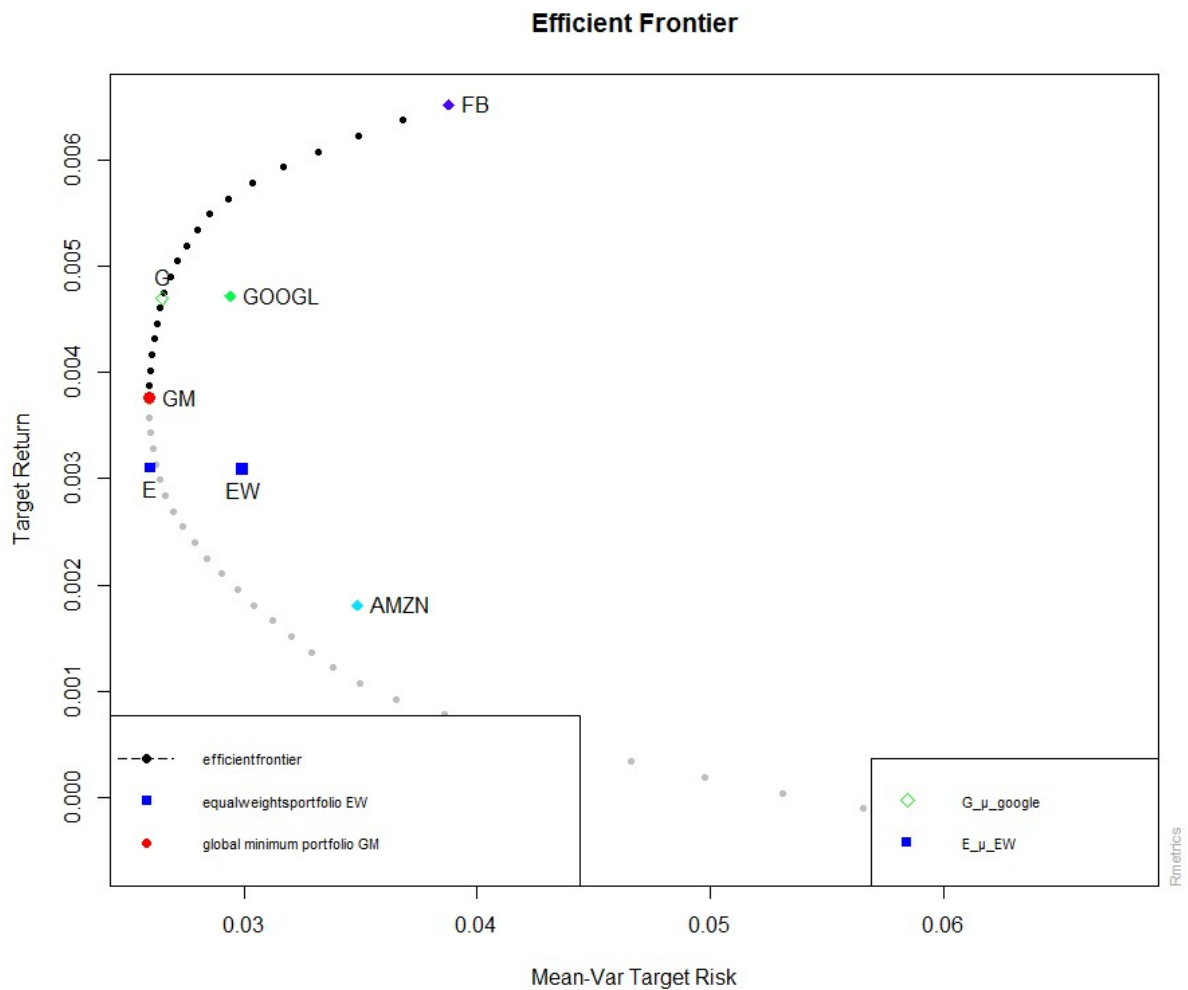
>text(sig.p.x, mu.p.x, labels="EW", pos=1)
> plot(longFrontier,1) # Plot EF
> plot(longFrontier,2) # Min. RiskPortfolio

```

```

> plot(longFrontier,4) # Risk/Return of Single Assets
> plot(longFrontier,5) # EqualWeightsPortfolio
>
> text(sig.gmin, mu.gmin, labels="GM", pos=4)
> text(sdfb, mu.vec["FB"], labels="FB", pos=4)
> text(sdgoogl, mu.vec["GOOGL"], labels="GOOGL", pos=4)
> text(sdamzn, mu.vec["AMZN"], labels="AMZN", pos=4)
> text(sig.p.x, mu.p.x, labels="EW", pos=1)
> text(0.026, 0.0031, labels="E", pos=1)
> text(0.0265, 0.0047, labels="G", pos=3)
> points(0.026, 0.0031, pch=15, col="blue")
> points(0.0265, 0.0047, pch=23, col="green")
>
>
> legend("bottomleft", c("efficientfrontier", "equalweightsportfolio", "global minimum
portfolio"), lty=c(5, NA, NA),
+       lwd=c(5, 0, 1), pch=c(15, 16, 15), col=c("black", "blue", "red"),
+       pt.cex=c(1, 1, 1), cex=0.7)
>
> legend("bottomright", c("G_μ_google", "E_μ_EW"), lty=c(NA, NA),
+       lwd=c(1, 1), pch=c(5, 15), col=c("green", "blue"), pt.cex=c(1, 1), cex=0.7)

```



**Figure 10:**Efficient Frontier\_LongOnly

Source: Authors creation (R-studio)

The black part of the concave line graphed in the figure 10 represents efficient frontier formed by the set of efficient portfolios. Minimum variance set has endpoints of relevant assets that

offer minimum or maximum expected return (FB or TESLA). Highest mean-return portfolio is plotted on the endpoint of efficient frontier (Facebook) while the lowest, Tesla, is plotted on the endpoint of the inefficient part. Tesla is not displayed in the graph because its low return would not allow for nice and accurate representation of other portfolios. Portfolio GM, portfolio G and portfolio FB are all considered efficient portfolios because no greater expected return can be found without increasing the corresponding risk. Portfolio GM is the global minimum variance portfolio or efficient portfolio that offers the lowest possible risk for any combination of assets. Portfolio FB or maximum return/maximum risk portfolio corresponds to 100% investment in the Facebook stock. Global minimum portfolio is not composed only of the least risky assets because of diversification and its risk reducing effects. Portfolio G has the same target return as GOOGL asset but for the reason that consists of investments in other assets and thereby taking advantage of diversification, it offers lower risk than portfolio GOOGLE. The same can be concluded for the portfolio E that has the same target return as portfolio EW but due to diversification it yields lower risk for the same expected return. Portfolios AMZN, EW and GOOGL are inefficient.

Comparing the frontier of a long-only constrained mean-variance portfolios (figure 10) with the frontier of unlimited short selling constrained mean-variance portfolios (figure 7 and figure 8) we notice that the first one is below the second (which allows short sales), meaning that for the same level of risk one achieves lower return.

## Chapter 7

# CVaR and Comparison with MV

### 7.1 Mean-CVaR Portfolio Computations in R

Mean-CVaR portfolio is defined as all others portfolios in Rmetrics by the time series data, portfolio specification and the constraints. Therefore, specifying these three components is the same as for the mean-variance portfolio specification. The difference between the CVaR and the MV approach is that in the case of the CVaR approach the type of portfolio has to be specified. The significance level of  $\alpha$  is set to 0,05 by default but can be changed by the user. Constraint is “LongOnly” meaning short selling is prohibited. It can be also modified by the user. Printout of specification is as follows:

```
> cvarSpec <- portfolioSpec()
> setType(cvarSpec) = "CVar"
> setAlpha(cvarSpec) = 0.05
> setSolver(cvarSpec) = "solveRglpk"
> print(cvarSpec)

Model List:
Type:          CVar
Optimize:      minRisk
Estimator:     covEstimator
Params:        alpha = 0.05

Portfolio List:
Target Weights: NULL
Target Return:  NULL
Target Risk:    NULL
Risk-Free Rate: 0
Number of Frontier Points: 50

Optim List:
Solver:         solveRglpk
Objective:      portfolioObjective portfolioReturn portfolioRisk
Trace:          FALSE
```

In following examples first we will compute equal weights feasible portfolio.

```
> portfolioReturns <- na.omit(ROC(portfolioPrices))
> portfolioReturns <- as.timeSeries(portfolioReturns)
> CvarData <- portfolioReturns
> ewPortfolio <- feasiblePortfolio(
+   data <- CvarData,
+   spec <- cvarSpec,
+   constraints <- "LongOnly")
> print(ewPortfolio)
```

```
Title:
CVAR Feasible Portfolio
Estimator:      covEstimator
Solver:         solveRglpk.CVAR
Optimize:       minRisk
Constraints:     LongOnly
```

```

Portfolio Weights:
  FB.Close  AMZN.Close  GOOGL.Close  TSLA.Close
    0.25      0.25      0.25      0.25

Covariance Risk Budgets:
  FB.Close  AMZN.Close  GOOGL.Close  TSLA.Close
    0.2389    0.2129    0.1307    0.4174

Target Returns and Risks:
  mean    Cov    CVaR    VaR
0.0031  0.0300  0.0721  0.0390

```

The target return is specified as the return from the equally weighted portfolio. Now we search for the portfolio with the same return but lower covariance risk.

```

> minriskSpec <- portfolioSpec()
> setType(minriskSpec) <- "cvar"
> setAlpha(minriskSpec) <- 0.05
> setSolver(minriskSpec) <- "solveRglpk.CVAR"
> setTargetReturn(minriskSpec) <- getTargetReturn(ewPortfolio@portfolio)["mean"]
> minriskPortfolio <- efficientPortfolio(data = CvarData, spec = minriskSpec,
+                                       constraints = "LongOnly")
> print(minriskPortfolio)

```

```

Title:
CVaR Efficient Portfolio
Estimator:      covEstimator
Solver:         solveRglpk.CVAR
Optimize:       minRisk
Constraints:     LongOnly
VaR Alpha:      0.05

Portfolio Weights:
  FB.Close  AMZN.Close  GOOGL.Close  TSLA.Close
    0.0000    0.1431    0.6327    0.2241

Covariance Risk Budgets:
  FB.Close  AMZN.Close  GOOGL.Close  TSLA.Close
    0.0000    0.1176    0.5262    0.3562

Target Returns and Risks:
  mean    Cov    CVaR    VaR
0.0031  0.0272  0.0576  0.0312

```

The covariance risk has dropped from 0,03 to 0,027 for the same target return and weights have been completely changed.

Next we compute the global minimum mean-CVaR portfolio, which is the efficient portfolio with the lowest possible risk.

```

> globminSpec <- portfolioSpec()
> setType(globminSpec) <- "cvar"
> setAlpha(globminSpec) <- 0.05
> setSolver(globminSpec) <- "solveRglpk.CVAR"
> setTargetReturn(globminSpec) <- getTargetReturn(ewPortfolio@portfolio)["mean"]
> globminPortfolio <- minriskPortfolio(data = CvarData, spec = globminSpec,
+                                       constraints = "LongOnly")
> print(globminPortfolio)

```

```

Title:

```



```

CvAR Minimum Risk Portfolio
Estimator:          covEstimator
Solver:             solveRglpk.CVAR
Optimize:           minRisk
Constraints:         LongOnly
VaR Alpha:          0.05

Portfolio Weights:
  FB.Close  AMZN.Close  GOOGL.Close  TSLA.Close
    0.0000    0.0000    0.8021    0.1979

Covariance Risk Budgets:
  FB.Close  AMZN.Close  GOOGL.Close  TSLA.Close
    0.0000    0.0000    0.7407    0.2593

Target Returns and Risks:

   mean   Cov   CVaR   VaR
0.0036 0.0273 0.0538 0.0411

```

R output 2

The portfolio is dominated by the low-risk Google stocks which contribute 80% to the weights of the optimized portfolio.

## 7.2 Mean-CVaR Portfolio Frontiers

In the following example long-only frontiers of CVaR and MV portfolios are computed. Output is shortened to 5 points starting with the portfolio with the lowest return and ending with the portfolio with the highest return. Since this is the case with restricted short sales, our frontier reaches both end of the efficient and inefficient part of the envelope. This will enable us to compare risk/return characteristics of the two optimization methods. Figures 11 and 12 show the results for the weights along the minimum variance locus and efficient frontier. Target returns are increasing from left to right and target risks are increasing both to the left and right with respect to the black separation line.

```

> longSpec <- portfolioSpec()
> setType(longSpec) <- "cvar"
> setAlpha(longSpec) <- 0.05
> setNFrontierPoints(longSpec) <- 5
> setSolver(longSpec) <- "solveRglpk.CVAR"
> longFrontier <- portfolioFrontier(data = CvARdata, spec = longSpec,
+                                   constraints = "LongOnly")
> par(mfrow = c(3, 1), mar = c(3.5, 4, 4, 3) + 0.1)
> weightsPlot(longFrontier)
> text <- "Mean-CVaR Portfolio - Long Only Constraints"
> mtext(text, side = 3, line = 3, font = 2, cex = 0.9)

```

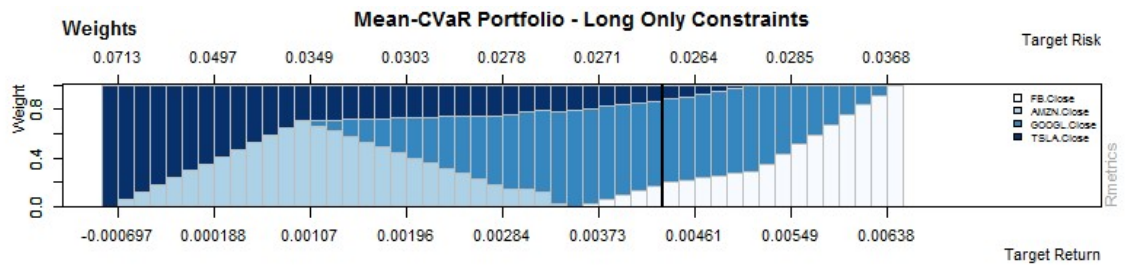


Figure 11:Weights along envelope set of mean-CVaR portfolios

Source: Authors creation (R-studio)

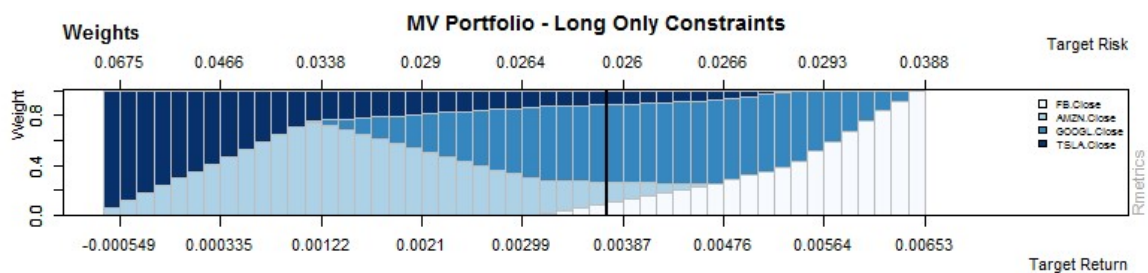


Figure 12: Weights along envelope set of MV portfolios

Source: Authors creation (R-studio)

The following example gives printout list of the weights, the target return and risk values along the envelope set for the CVaR portfolio, starting with the portfolio with the lowest return and ending with the highest return reached.

```
> longSpec <- portfoliospec()
> setType(longSpec) <- "Cvar"
> setAlpha(longSpec) <- 0.05
> setNFrontierPoints(longSpec) <- 5
> setSolver(longSpec) <- "solveRglpk.CVAR"
> longFrontier <- portfolioFrontier(data = cvarData, spec = longSpec,
+                                 constraints = "LongOnly")
> print(longFrontier)
```

```
Title:
Cvar Portfolio Frontier
Estimator: covEstimator
Solver: solveRglpk.CVAR
Optimize: minRisk
Constraints: LongOnly
Portfolio Points: 5 of 5
VaR Alpha: 0.05
```

```
Portfolio Weights:
  FB.Close AMZN.Close GOOGL.Close TSLA.Close
1  0.0000  0.0000  0.0000  1.0000
2  0.0000  0.6965  0.0114  0.2921
3  0.0000  0.1593  0.5931  0.2476
4  0.2305  0.0000  0.6927  0.0768
```

```
5 1.0000 0.0000 0.0000 0.0000
```

Covariance Risk Budgets:

```
FB.Close AMZN.Close GOOGL.Close TSLA.Close
1 0.0000 0.0000 0.0000 1.0000
2 0.0000 0.5647 0.0035 0.4318
3 0.0000 0.1275 0.4533 0.4192
4 0.2441 0.0000 0.7033 0.0526
5 1.0000 0.0000 0.0000 0.0000
```

Target Returns and Risks:

```
mean Cov CVaR VaR
1 -0.0007 0.0713 0.1421 0.1342
2 0.0011 0.0347 0.0826 0.0447
3 0.0029 0.0277 0.0598 0.0326
4 0.0047 0.0266 0.0560 0.0384
4 0.0065 0.0388 0.0944 0.0552
```

The following example gives printout list of the weights, the target return and risk values along the envelope set for the MV Portfolio, starting with the portfolio with the lowest return and ending with the highest reached return.

```
> print(frontier)
```

Title:

```
MV Portfolio Frontier
Estimator: covEstimator
Solver: solveRquadprog
Optimize: minRisk
Constraints: LongOnly
Portfolio Points: 5 of 24
```

Portfolio weights:

```
FB.Close AMZN.Close GOOGL.Close TSLA.Close
1 0.0000 0.1202 0.0000 0.8798
6 0.0000 0.7212 0.0000 0.2788
12 0.0000 0.3470 0.5063 0.1467
18 0.2485 0.0000 0.6686 0.0828
24 1.0000 0.0000 0.0000 0.0000
```

Covariance Risk Budgets:

```
FB.Close AMZN.Close GOOGL.Close TSLA.Close
1 0.0000 0.0154 0.0000 0.9846
6 0.0000 0.5989 0.0000 0.4011
12 0.0000 0.3727 0.4451 0.1822
18 0.2698 0.0000 0.6686 0.0616
24 1.0000 0.0000 0.0000 0.0000
```

Target Returns and Risks:

```
mean Cov CVaR VaR
1 -0.0004 0.0636 0.1280 0.1181
6 0.0011 0.0346 0.0826 0.0446
12 0.0029 0.0265 0.0617 0.0367
18 0.0047 0.0265 0.0561 0.0358
24 0.0065 0.0388 0.0944 0.0552
```

### **7.3 Comparison of relative Performance of MV and CVaR Portfolios before the Corona Crisis**

In this section we present the results of the two methods under evaluation for the period before the corona crisis (from December 2018 to December 2019). This period should capture positive financial situations on the market. Comparing the portfolio sets that resulted from MV and CVaR methods (R output 1 and R output 2) it can be concluded that MV portfolio method yielded a better rate of return with lower risk. These results refer to the whole period under consideration.

Figure 12 presents weights along the envelope frontier in the case of the MV and Figure 11 shows weights in the case of CVaR optimization. Comparing the weights obtained by both optimization methods, we can observe the difference in composition of portfolios in the middle part or in the part near separation line that marks the position between the minimum variance locus and efficient frontier.

The result shows that the weights of Amazon stocks are lower and weights of Google are higher for the portfolios constructed by minimizing CVaR. The reason for this could be that the distribution for the returns of the Amazon exhibits a negative skewness and has fatter tails than the returns of the other stocks. Negative skewness shows frequent small gains and a few extreme losses. This can lead to an underestimation of the risk connected with Amazon stocks. This aspect hasn't been taken into consideration when choosing weights according to the mean-variance criteria but it is captured in CVaR optimization method. In the end the underestimation of the risk leads to an overestimation of the optimal weights.

### **7.4 Comparison of relative Performance of MV and CVaR Portfolios during the Corona Crisis**

In this section we will compute feasible portfolios, portfolios with the lowest risk for a given target return and global minimum-risk portfolios using both optimization methods, MV and CVaR. Period under consideration is the interval from December 2019 to July 2020. This period captures the outbreak of the corona virus. MV and CVaR global minimum-risk portfolios will be compared and analyzed.

We will start the analysis with the summary and plots of asset prices (presented in Figure 13) and calculation of the skewness of returns. The plots for the period influenced by the corona virus (Figure 13) unveil volatility, big jumps and drops in the prices (taking into account the absolute price difference). In one month Google stock had plunge from 1519 to 1068 dollars and Tesla stock from ca. 900 to ca. 400 dollars. Hence we can expect that these two stocks will constitute a very little, if any portion of the optimized portfolios. Moreover, Tesla has the greatest negative skewness (-1,09) indicating that the values of returns are more likely to be below the mean, which is an additional reason to classify this stock as risky asset.

```
plot(portfolioPrices[, 1], main = "Time series plots FB", col = "steelblue")
> plot(portfolioPrices[, 2], main = "Time series plots AMZN", col = "steelblue")
> plot(portfolioPrices[, 3], main = "Time series plots GOOGL", col = "steelblue")
> plot(portfolioPrices[, 4], main = "Time series plots TSLA", col = "steelblue")

> summary(portfolioPrices_corona)
      Index      FB.Close      AMZN.Close      GOOGL.Close      TSLA.Close
Min. :2019-12-02 Min. :149.7 Min. :1752 Min. :1068 Min. : 335.9
1st Qu.:2020-01-23 1st Qu.:191.3 1st Qu.:1872 1st Qu.:1307 1st Qu.: 495.3
Median :2020-03-16 Median :208.7 Median :2043 Median :1363 Median : 701.3
Mean :2020-03-16 Mean :202.9 Mean :2133 Mean :1352 Mean : 672.6
3rd Qu.:2020-05-07 3rd Qu.:218.0 3rd Qu.:2410 3rd Qu.:1431 3rd Qu.: 818.1
Max. :2020-06-29 Max. :238.8 Max. :2759 Max. :1519 Max. :1079.8

> kurtosis(portfolioReturns_corona)
      FB.Close AMZN.Close GOOGL.Close TSLA.Close
Excess Kurtosis -0.3275306  2.871792  1.258408  1.213307

> skewness(portfolioReturns_corona)
      FB.Close AMZN.Close GOOGL.Close TSLA.Close
Skewness -0.2341921  0.122417 -0.808046 -1.088944
```

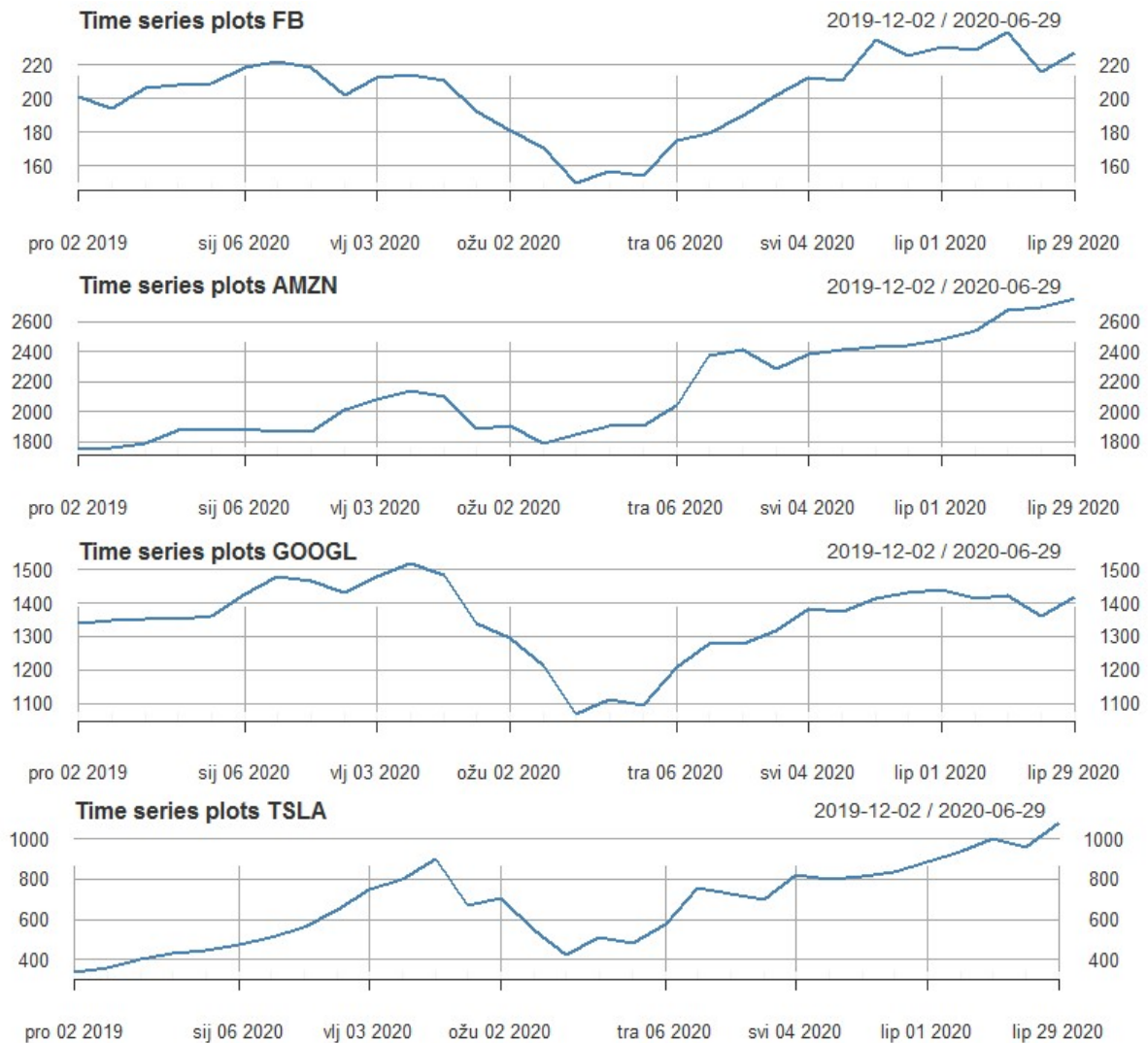


Figure 13: Time series plots of stock prices for the period from December 2019 to July 2020

Source: Authors creation (R-studio)

The following examples present portfolio computations. First, we compute equal weights feasible portfolio with “LongOnly” constraint, using CVaR and then MV parameters.

```
> cvarSpec_corona <- portfolioSpec()
> setType(cvarSpec_corona) <- "CVAR"
> nAssets <- ncol(Cvardata)
> setWeights(cvarSpec_corona) <- rep(1/nAssets, times = nAssets)
> setSolver(cvarSpec_corona) <- "solveRglpk.CVAR"
> nAssets <- ncol(portfolioReturns_corona)
> ewPortfolio <- feasiblePortfolio(
+   data <- portfolioReturns_corona,
+   spec <- cvarSpec_corona,
+   constraints <- "LongOnly")
> print(ewPortfolio)
```

```
Title:
CVAR Feasible Portfolio
Estimator: covEstimator
Solver: solveRglpk.CVAR
Optimize: minRisk
Constraints: LongOnly
```

Portfolio weights:

```

      FB.Close  AMZN.Close  GOOGL.Close  TSLA.Close
      0.25      0.25      0.25      0.25
Covariance Risk Budgets:
      FB.Close  AMZN.Close  GOOGL.Close  TSLA.Close
      0.1895    0.1317    0.1745    0.5042
Target Returns and Risks:
      mean    Cov    CVaR    VaR
0.0150 0.0604 0.1384 0.1172

> ewSpec <- portfolioSpec()
> nAssets <- ncol(portfolioReturns_corona)
> setWeights(ewSpec) <- rep(1/nAssets, times = nAssets)
> ewPortfolio_corona <- feasiblePortfolio(
+   data = portfolioReturns_corona,
+   spec = ewSpec,
+   constraints = "LongOnly")
> print(ewPortfolio_corona)

Title:
MV Feasible Portfolio
Estimator:      covEstimator
Solver:         solveRquadprog
Optimize:       minRisk
Constraints:     LongOnly

Portfolio Weights:
      FB.Close  AMZN.Close  GOOGL.Close  TSLA.Close
      0.25      0.25      0.25      0.25
Covariance Risk Budgets:
      FB.Close  AMZN.Close  GOOGL.Close  TSLA.Close
      0.1895    0.1317    0.1745    0.5042
Target Returns and Risks:
      mean    Cov    CVaR    VaR
0.0150 0.0604 0.1384 0.1172

```

Secondly, specifying the target return as the return of the equally weighted portfolio, we compute MV and CVaR portfolios with the same return but lower risk.

```

> minriskSpec <- portfolioSpec()
> setType(minriskSpec) <- "CVaR"
> setAlpha(minriskSpec) <- 0.05
> setSolver(minriskSpec) <- "solveRglpk.CVAR"
> setTargetReturn(minriskSpec) <- getTargetReturn(ewPortfolio@portfolio)["mean"]
> minriskPortfolio <- efficientPortfolio(data = portfolioReturns_corona, spec = minriskSpec,
+   constraints = "LongOnly")
> print(minriskPortfolio)

Title:
CVaR Efficient Portfolio
Estimator:      covEstimator
Solver:         solveRglpk.CVAR
Optimize:       minRisk
Constraints:     LongOnly
VaR Alpha:      0.05

Portfolio Weights:
      FB.Close  AMZN.Close  GOOGL.Close  TSLA.Close
      0.0122    0.9878    0.0000    0.0000
Covariance Risk Budgets:
      FB.Close  AMZN.Close  GOOGL.Close  TSLA.Close
      0.0048    0.9952    0.0000    0.0000
Target Returns and Risks:
      mean    Cov    CVaR    VaR
0.0150 0.0441 0.0920 0.0630

> minriskSpec <- portfolioSpec()
> minriskSpec <- portfolioSpec()
> targetReturn <- getTargetReturn(ewPortfolio_corona@portfolio)["mean"]
> setTargetReturn(minriskSpec) <- targetReturn
> minriskPortfolio_corona <- efficientPortfolio(
+   data = portfolioReturns_corona,
+   spec = minriskSpec,
+   constraints = "LongOnly")
> print(minriskPortfolio_corona)

Title:
MV Efficient Portfolio
Estimator:      covEstimator
Solver:         solveRquadprog
Optimize:       minRisk
Constraints:     LongOnly

Portfolio Weights:

```

```

      FB.Close  AMZN.Close  GOOGL.Close  TSLA.Close
      0.0430    0.9426    0.0000    0.0143

Covariance Risk Budgets:
      FB.Close  AMZN.Close  GOOGL.Close  TSLA.Close
      0.0201    0.9489    0.0000    0.0311

Target Returns and Risks:
      mean    Cov    CVaR    VaR
0.0150  0.0441  0.0944  0.0657

```

Finally, setting parameters for global minimum-risk portfolio we get the MV and CVaR efficient portfolios with the lowest possible risk.

```

> globminSpec <- portfolioSpec()
> globminSpec_corona <- portfolioSpec()
> setType(globminSpec_corona) <- "CVaR"
> setAlpha(globminSpec_corona) <- 0.05
> setSolver(globminSpec_corona) <- "solveRglpk.CVAR"
> setTargetReturn(globminSpec_corona) <- getTargetReturn(ewPortfolio@portfolio)["mean"]
> globminPortfolio_corona <- minriskPortfolio(data = portfolioReturns_corona, spec = globminSpec_corona,
+ constraints = "LongOnly")
> print(globminPortfolio_corona)

Title:
CVaR Minimum Risk Portfolio
Estimator:      covEstimator
Solver:         solveRglpk.CVAR
Optimize:       minRisk
Constraints:     LongOnly
var Alpha:      0.05

Portfolio Weights:
      FB.Close  AMZN.Close  GOOGL.Close  TSLA.Close
      0.5904    0.4096    0.0000    0.0000

Covariance Risk Budgets:
      FB.Close  AMZN.Close  GOOGL.Close  TSLA.Close
      0.7394    0.2606    0.0000    0.0000

Target Returns and Risks:
      mean    Cov    CVaR    VaR
0.0086  0.0442  0.0845  0.0621


> globminSpec_cor <- portfolioSpec()
> globminPortfolio_cor <- minvariancePortfolio(
+ data = portfolioReturns_corona,
+ spec = globminSpec_cor,
+ constraints = "LongOnly")
> print(globminPortfolio_cor)

Title:
MV Minimum Variance Portfolio
Estimator:      covEstimator
Solver:         solveRquadprog
Optimize:       minRisk
Constraints:     LongOnly

Portfolio Weights:
      FB.Close  AMZN.Close  GOOGL.Close  TSLA.Close
      0.0309    0.5439    0.4253    0.0000

Covariance Risk Budgets:
      FB.Close  AMZN.Close  GOOGL.Close  TSLA.Close
      0.0309    0.5439    0.4253    0.0000

Target Returns and Risks:
      mean    Cov    CVaR    VaR
0.0092  0.0388  0.0908  0.0638

```

The optimization shows that the performance and risk measure for each global minimum-risk portfolio differs. First, the composition of the two portfolios reveals that CVaR optimization method takes into account severe losses whereas MV does not. As we have seen, Tesla has the greatest negative skewness (-1,09) followed by Google (-0,8) and this is the reason why portfolio under CVaR optimization does not include this asset. However, MV optimized portfolio contains Google stock, actually in a pretty big proportion (0,4253), which suggests that MV does not take into account the skewness of the losses when forming the optimal



portfolio and therefore can lead to severe losses. Moreover, CVaR portfolio has a lower return (0,0086) but a lower risk (0,0621) than MV portfolio (0,0092; 0.638). We can conclude that MV outperforms CVaR in terms of return but higher return comes at price of higher risk.

## 7.5 Comparison of CVaR and MV Portfolios during both Periods

We start this analysis with visual representation (Figure 14) of stock returns before and during the corona crisis. Movements of stock prices are usually unpredictable even during the stable periods, meaning that uncertainty of financial assets exhibits a random walk in the price behavior. However, during the crisis stock prices should reveal overall volatility and huge drops. Figures 13 and 14 confirm that overall volatility dramatically scaled up in March 2020 at time of the outbreak of the corona crisis. However, this abrupt price drops lasted only for a short time. Jumps and drops thereafter were more frequent and inconsistent compared to the period before the corona crisis.

```
> colnames(portfolioReturns)
[1] "FB.Close" "AMZN.Close" "GOOGL.Close" "TSLA.Close"
> plot(portfolioReturns, main = "Time series plots", col = "steelblue")
```

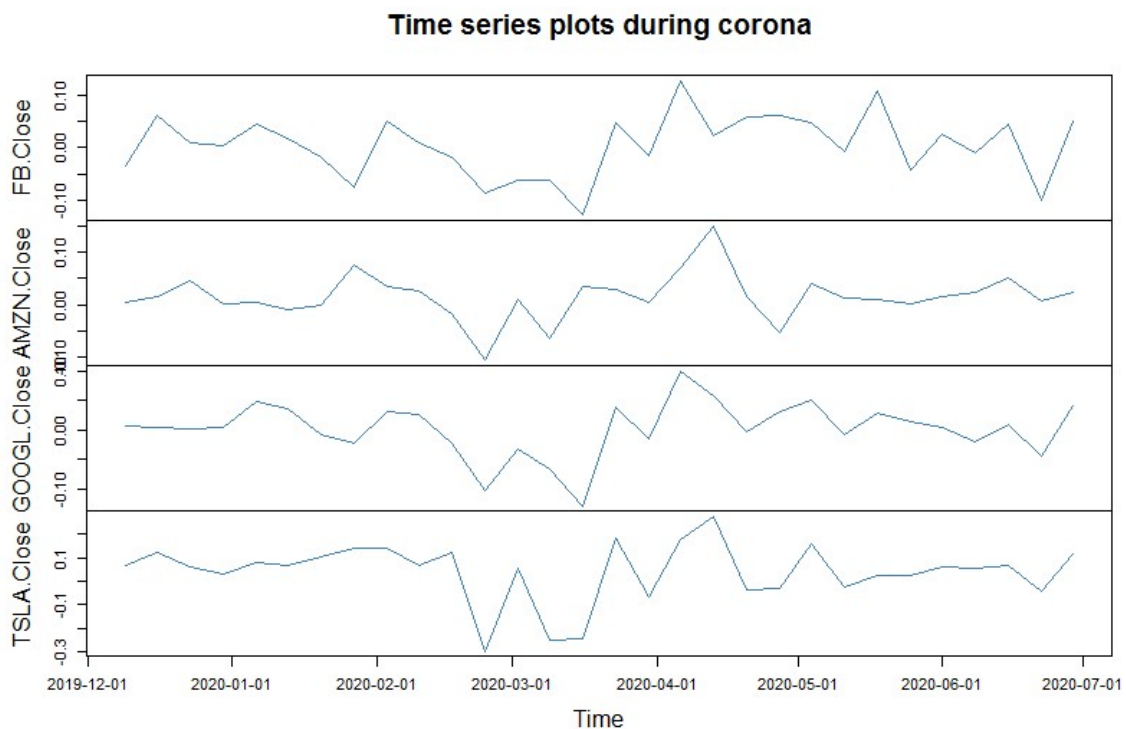


Figure 14: Time series plot of stock returns in the period from December 2019 to July 2020

Source: Authors creation (R-studio)

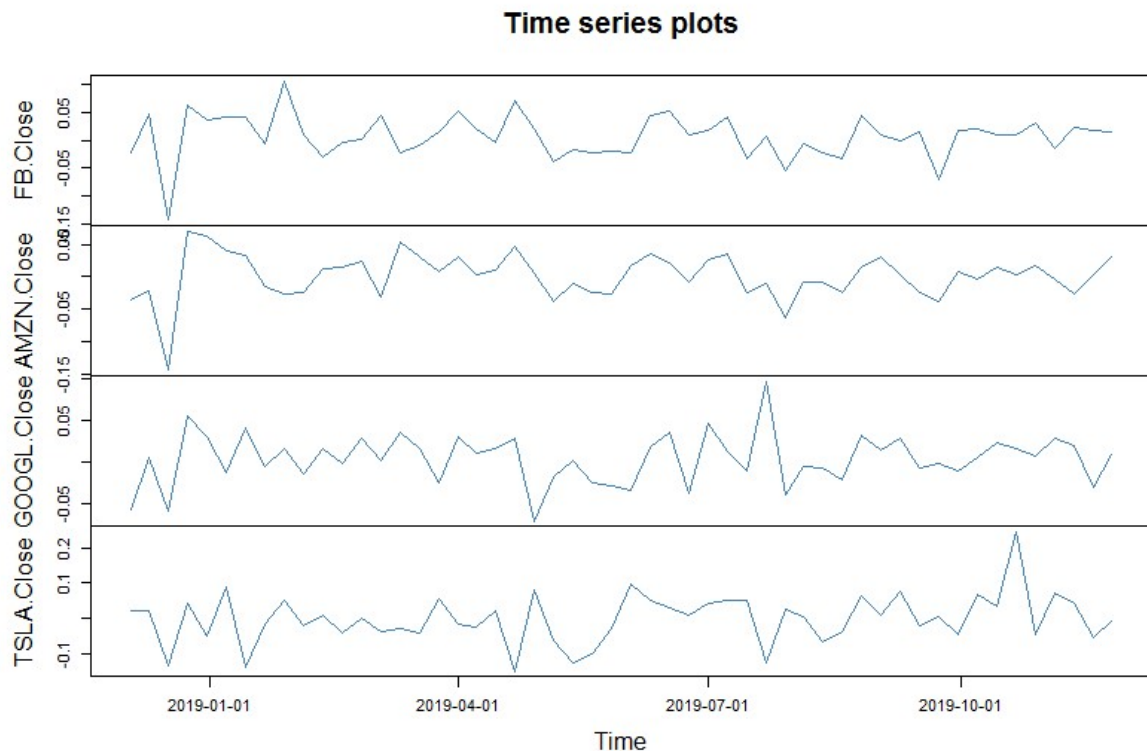


Figure 15: Time series plot of stock returns in the period from December 2018 to July 2019

Source: Authors creation (R-studio)

Before we analyze MV and CVaR optimized portfolios and compare the two periods, we will make short comments on skewness (R output 3) that will help us better understand the composition of the optimized portfolios.

```
> skewness(portfolioReturns)
      FB.Close AMZN.Close GOOGL.Close TSLA.Close
Skewness -0.7878122 -1.182657  0.0009393663  0.2948031

> skewness(portfolioReturns_corona)
      FB.Close AMZN.Close GOOGL.Close TSLA.Close
Skewness -0.2341921  0.122417  -0.808046  -1.088944

> kurtosis(portfolioReturns)
      FB.Close AMZN.Close GOOGL.Close TSLA.Close
Excess Kurtosis 2.885864  4.088024  0.7807338  1.564074

> kurtosis(portfolioReturns_corona)
      FB.Close AMZN.Close GOOGL.Close TSLA.Close
Excess Kurtosis -0.3275306  2.871792  1.258408  1.213307
```

R output 3

Several studies in empirical finance confirm that distributions of asset returns are characterized by negative skewness and excess kurtosis, meaning that the assumption of normal distribution

is continuously being violated (Beedles, 1986<sup>29</sup> Lux & Marchesi, 2000<sup>30</sup>). From looking at the skewness before the crisis, it can be seen that Google was near zero (0,0009), exhibiting normal symmetric distribution, and Tesla was slightly positive (0,294) meaning that it had frequent small losses and a few extreme gains. Amazon had negative skewness (-1,18), as well as Facebook (-0,787). CVaR optimized global minimum-risk portfolio was constituted according to these parameters:

```
Portfolio Weights:
  FB.Close  AMZN.Close  GOOGL.Close  TSLA.Close
    0.0000      0.0000      0.8021      0.1979
```

On the other hand, MV optimized portfolio included Facebook and Amazon stocks despite their risky component:

```
Portfolio weights:
  FB  AMZN  GOOGL  TSLA
0.0907 0.1754 0.6205 0.1134
```

Moreover, Amazon had very high kurtosis (4,08) and that is the reason why it is not included in CVaR portfolio.

Additionally, it can be concluded that during the corona crisis both MV and CVaR optimized portfolios yielded higher return but also higher risk, where CVaR had a slightly lower return than MV. The reason for this could be in a shorter time interval under consideration during which there was one big drop for all stocks but also some jumps. This can be seen from lower kurtosis of stocks, where portfolio with smaller kurtosis tends to have less extreme events and is preferred by investors. What is the most evident is that the traditional MV model deals with the variance of returns as risk measure and determines optimal portfolio accordingly, without incorporating extreme left-tail events as CVaR model does. Therefore, optimized MV and CVaR portfolios have pretty different compositions of assets. That can lead to underperformance of MV portfolio.

---

<sup>29</sup>Beedles, W., 1986. Asymmetry in Australian Equity Returns. *Australian Journal of Management*, 11(1), pp.1-12

<sup>30</sup>LUX, T. and MARCHESI, M., 2000. VOLATILITY CLUSTERING IN FINANCIAL MARKETS: A MICROSIMULATION OF INTERACTING AGENTS. *International Journal of Theoretical and Applied Finance*, 03(04), pp.675-702..

## Conclusion

Portfolio theory is one of the most widely studied parts of quantitative investment analysis. The main problem each investor faces is to decide in which securities to invest and then select portfolio to invest in from the set of all feasible portfolios. Harry Markowitz proposed solution to this problem referred to as the „portfolio selection problem“. Markowitz's theory provides the theoretical foundation for evaluating portfolios on the basis of their expected returns, seen as a measure of potential reward, and risk as measured by the standard deviation. He first derived the concept of an efficient portfolio as one having the smallest risk for a given level of expected return. He also illustrated the principle of diversification, that is the process of selecting assets that are differently affected by specific market volatility, as a very important concept of risk reduction. One main characteristic of diversification is making balance between the risk and rewards of distinct investments. The Markowitz model yields efficient portfolios assuming that we have perfect information on the expected returns and covariances for the assets that we are considering. In reality, we do not have perfect information and small changes in time series from the past returns lead to changes in the expected returns that can distort the result of efficient portfolios. Markowitz model uses variance as a measure of risk with the assumption of normal distribution of returns. Recent history has shown that financial assets do not exhibit normal distribution and Markowitz's assumption of normally distributed asset returns makes optimisation process open to underestimation of risk. Due to this drawbacks, some downside risk measures have been introduced, such as CvaR. In this thesis CvaR was implemented and tested in R software together with MV optimization method. The portfolio optimization was performed for two distinct time periods. The first interval covers the period from December 2018 to December 2019, while the second interval covers the period during the corona virus outbreak, that is from December 2019 to July 2020. Periods were chosen as to capture both favourable and negative financial market conditions. Results show that MV global minimum-risk portfolio outperformed CvaR global minimum-risk portfolio in terms of rate of return for both periods. Also, difference in rates of return was much smaller during the period before the corona virus crisis. Shorter period of historical observations during the corona virus crisis could have had impact on higher, not so realistic, rates of return. Moreover, better performance of MV method might be due to the MV concept that penalizes losses and gains symmetrically. Both MV and CvaR optimized portfolios had greater risk during second period. This result was

expected due to the uncertainty that crisis has brought. The important conclusion from this study is that there is a difference in how these two methods measure and evaluate risk. Unlike in CVaR method, the presence of negative skewness, higher probability of getting negative returns and excess kurtosis, indicating higher probability of extreme events, are not included in MV framework. This has led to different compositions of MV and CVaR portfolios. Hence, nowadays when financial markets exhibit many abrupt changes, portfolio optimisation based on MV framework can lead to underestimation of risk, yielding inefficient portfolio rather than the efficient one. Further analysis can be done in revising the traditional MV approach by including other moments of risk.

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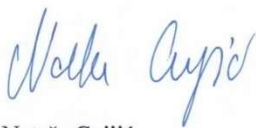
Name and family name of student

NATAŠA CVIJIC

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Zagreb, 11.07.2020.



Nataša Cvijic